Minnesota State High School Mathematics League

2020-21 Meet 2, Individual Event A

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

__________ 1. Gerry has 20 coins, all dimes and quarters, totaling $\$2.75$. How many dimes does Gerry have?

__________ 2. Find the sum of all the possible values for x, such that $|\, x-5 \,| = 7-2x$.

$a+b=$ __________ 3. All the possible values for x, such that $|\, 2x+5 \,| \leq x+7$, can be written as an interval $[a,\, b]$. Determine the value of $a+b$.

__________ 4. Let m be a positive integer and let the lines $13x+11y=700$ and $y=mx-1$ intersect at a point with integer coordinates. Determine the **sum** of all possible values for m.

Name: ________________ Team: ________________

Minnesota State High School Mathematics League

2020-21 Meet 2, Individual Event A

SOLUTIONS

$\boxed{15}$

1. Gerry has 20 coins, all dimes and quarters, totaling $\$2.75$. How many dimes does Gerry have?

Let $d = \#$ *dimes, then* $\#$ *quarters* $= 20 - d$. *So* $10d + 25(20-d) = 275 \Rightarrow$
$10d + 500 - 25d = 275 \Rightarrow -15d = -225 \Rightarrow d = 15$.

$\boxed{2}$

2. Find the sum of all the possible values for x, such that $|\ x-5\ | = 7-2x$.

Because $|a| \geq 0$, $7-2x \geq 0 \Rightarrow x \leq 3\frac{1}{2}$. *But if* $x \leq 3\frac{1}{2}$, *then* $x-5<0$. *So*
$|x-5| = -x+5$. *Solving,* $-x+5 = 7-2x \Rightarrow x = 2$, *an acceptable solution.*

$\boxed{-2}$

3. All the possible values for x, such that $|\ 2x+5\ | \leq x+7$, can be written as an interval $[a,\ b]$. Determine the value of $a+b$.

Observe $x \geq -7$. *Case 1:* $x \geq -\frac{5}{2}$, *then* $2x+5 \leq x+7 \Rightarrow x \leq 2$. *Therefore,*
$-\frac{5}{2} \leq x \leq 2$ *work. Case 2:* $x < -\frac{5}{2}$, *then* $-2x-5 \leq x+7 \Rightarrow -12 \leq 3x \Rightarrow x \geq -4$.
Therefore, $-4 \leq x < -\frac{5}{2}$ *work. Combining the two cases yields:* $[-4,\ 2]$.

$\boxed{6}$

4. Let m be a positive integer and let the lines $13x + 11y = 700$ and $y = mx - 1$ intersect at a point with integer coordinates. Determine the **sum** of all possible values for m.

$13x + 11(mx-1) = 700 \Rightarrow x = \dfrac{711}{13+11m}$. *Let* $d = 13 + 11m$. d *must divide* $711 = 3^2 \cdot 79$.
Therefore, d *must be* $1,\ 3,\ 9,\ 79,\ 3\cdot 79,$ *or* $9 \cdot 79$. *But* $d > 13$, *so there are only 3 cases to check: a)* $13 + 11m = 79 \Rightarrow 11m = 66 \Rightarrow m = 6$, *b)* $13 + 11m = 3\cdot 79 \Rightarrow 11m = 224$, *but* 224 *is not divisible by 11, c)* $13 + 11m = 9\cdot 79 \Rightarrow 11m = 698$, *but* 698 *is not divisible by 11.*

Minnesota State High School Mathematics League

2020-21 Meet 2, Individual Event B

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points.
Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

$h =$ ______ in.

1. A pyramid with a square base has a base edge of 6 in. and a volume of 36 cu.in. Determine the height of the pyramid in inches.

$CF =$ ______

2. In *Figure 2*, in $\triangle ABC$, $AB = 17$, $BC = 21$, and $AC = 30$. The three cevians, $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ intersect at G. If $\dfrac{AD}{DB} = \dfrac{4}{7}$ and $\dfrac{BE}{EC} = \dfrac{3}{4}$, determine length CF.

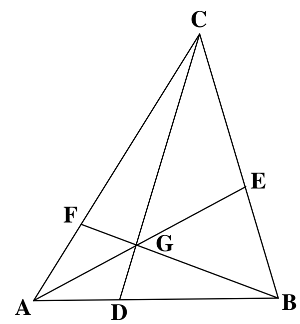


Figure 2

$a + b =$ ______

3. In $\triangle ABC$, the bisector of $\angle A$ intersects $\overline{BC}$ at D. If $AD = 21\sqrt{2}$, $AB = 20\sqrt{3}$, and $\angle ADB \cong \angle BAC$, the length of $\overline{CD}$ can be written as $a\sqrt{b}$, where b is square-free. Determine the value of $a + b$.

______

4. In $\triangle ABC$, $AB = 12$, $BC = 14$, and $AC = 20$. Determine the **sum** of the **squares** of its three medians.

Name: ______________________ Team: ______________________

Minnesota State High School Mathematics League

2020-21 Meet 2, Individual Event B

SOLUTIONS

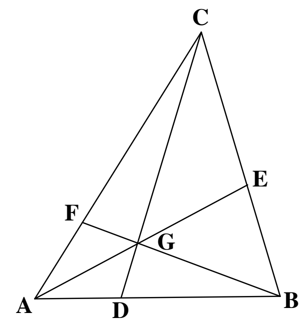
$\boxed{3}$

1. A pyramid with a square base has a base edge of 6 in. and a volume of 36 cu.in. Determine the height of the pyramid in inches.

$$\frac{1}{3}\left(6^2\right)(h) = 36 \Rightarrow 12h = 36 \Rightarrow h = 3.$$

$\boxed{21}$

2. In *Figure 2*, in $\triangle ABC$, $AB = 17$, $BC = 21$, and $AC = 30$. The three cevians, $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ intersect at G. If $\dfrac{AD}{DB} = \dfrac{4}{7}$ and $\dfrac{BE}{EC} = \dfrac{3}{4}$, determine length CF.



By Ceva's Theorem, $\dfrac{AD}{DB}\cdot\dfrac{BE}{EC}\cdot\dfrac{CF}{FA} = 1$. *So* $\dfrac{CF}{FA} = \dfrac{7}{3}$. *Since* $AC = 30$, $CF = 21$.

$\boxed{23}$

3. In $\triangle ABC$, the bisector of $\angle A$ intersects $\overline{BC}$ at D. If $AD = 21\sqrt{2}$, $AB = 20\sqrt{3}$, and $\angle ADB \cong \angle BAC$, the length of $\overline{CD}$ can be written as $a\sqrt{b}$, where b is square-free. Determine the value of $a + b$.

In Figure 3.1, let $m\angle BAD = m\angle DAC = x$. *Then* $m\angle BDA = 2x$. *By the Exterior Angle Theorem,* $m\angle BCA = x$. *So* $\triangle ADC$ *is isosceles and* $AD = CD = 21\sqrt{2}$.

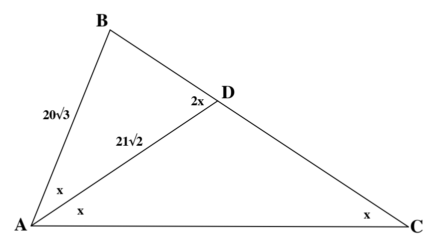


Figure 3.1

$\boxed{555}$

4. In $\triangle ABC$, $AB = 12$, $BC = 14$, and $AC = 20$. Determine the **sum** of the **squares** of its three medians.

To find x, *the length of median* $\overline{CM}$ *in* $\triangle ABC$, *Stewart's Formula would be:*

$$a^2\left(\frac{c}{2}\right) + b^2\left(\frac{c}{2}\right) = x^2c + c\left(\frac{c}{2}\right)\left(\frac{c}{2}\right) \Rightarrow x^2 = \frac{2a^2 + 2b^2 - c^2}{4}.$$ *The other two medians will be similar:*

$$y^2 = \frac{2a^2 + 2c^2 - b^2}{4} \text{ and } z^2 = \frac{2b^2 + 2c^2 - a^2}{4}.$$ *So*

$$x^2 + y^2 + z^2 = \frac{3a^2 + 3b^2 + 3c^2}{4} = \frac{3}{4}\left(12^2 + 14^2 + 20^2\right) = 555.$$

Minnesota State High School Mathematics League

2020-21 Meet 2, Individual Event C

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

$p+q=$ ____________

1. The value of $\sqrt{\dfrac{1+\cos 120^\circ}{2}} = \dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$p+q=$ ____________

2. If $\cos x = \dfrac{1}{4}$, the value of $\dfrac{\sin(4x)}{\sin x} = \dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$a+b=$ ____________

3. In the interval $0 \le x < 2\pi$, $\cos x \le \sin\left(\dfrac{x}{2}\right)$, when x is in the interval $[a\pi, b\pi]$. Determine the value of $a+b$ for the largest possible such interval.

$a+b=$ ____________

4. Let x and y be acute angles with $x > y$. If $\sin x + \sin y = 1$ and $\cos x + \cos y = 1.5$, the positive value of $\sin(x-y)$ can be written as $\dfrac{\sqrt{a}}{b}$, where a is square-free. Determine the value of $a+b$.

Name: ____________________ Team: ____________________

Minnesota State High School Mathematics League

2020-21 Meet 2, Individual Event C

SOLUTIONS

3

1. The value of $\sqrt{\dfrac{1+\cos 120^\circ}{2}} = \dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$\cos\left(\dfrac{x}{2}\right) = \pm\sqrt{\dfrac{1+\cos x}{2}}$. *So let* $x = 120$, $\cos\left(\dfrac{120^\circ}{2}\right) = \cos 60^\circ = \dfrac{1}{2}$.

1

2. If $\cos x = \dfrac{1}{4}$, the value of $\dfrac{\sin(4x)}{\sin x} = \dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$$\frac{\sin(4x)}{\sin x} = \frac{2\sin(2x)\cos(2x)}{\sin x} = \frac{2(2\sin x\cos x)(2\cos^2 x - 1)}{\sin x} = 4\left(\frac{1}{4}\right)\left(2\left(\frac{1}{4}\right)^2 - 1\right) = \frac{-7}{8}.$$

2

3. In the interval $0 \le x < 2\pi$, $\cos x \le \sin\left(\dfrac{x}{2}\right)$, when x is in the interval $[a\pi, b\pi]$. Determine the value of $a+b$ for the largest possible such interval.

Consider $\cos x \le \pm\sqrt{\dfrac{1-\cos x}{2}} \Rightarrow 2\cos^2 x + \cos x - 1 \le 0 \Rightarrow (2\cos x - 1)(\cos x + 1) \le 0$. *Its roots are* $\cos x = -1$ *and* $\cos x = \dfrac{1}{2}$, *which occur when* $x = \dfrac{3\pi}{2}$, $x = \dfrac{\pi}{3}$, *and* $x = \dfrac{5\pi}{3}$. *It is nonpositive between the two extreme values. So a solution happens when x is in the interval* $\left[\dfrac{\pi}{3}, \dfrac{5\pi}{3}\right]$. *So* $\dfrac{1}{3} + \dfrac{5}{3} = \dfrac{6}{3} = 2$.

47

4. Let x and y be acute angles with $x > y$. If $\sin x + \sin y = 1$ and $\cos x + \cos y = 1.5$, the positive value of $\sin(x-y)$ can be written as $\dfrac{\sqrt{a}}{b}$, where a is square-free. Determine the value of $a+b$.

Squaring the two equations, yields $\sin^2 x + 2\sin x\sin y + \sin^2 y = 1$ *and* $\cos^2 x + 2\cos x\cos y + \cos^2 y = \dfrac{9}{4}$. *Adding these and simplifying, yields* $\cos x\cos y + \sin x\sin y = \dfrac{5}{8}$, *implying* $\cos(x-y) = \dfrac{5}{8}$.

$\sin(x-y) = \sqrt{1-\cos^2(x-y)} = \sqrt{1-\left(\frac{5}{8}\right)^2} = \dfrac{\sqrt{39}}{8}$.

Minnesota State High School Mathematics League

2020-21 Meet 2, Individual Event D

Question #1 is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 15 minutes for this event.

NO CALCULATORS are allowed on this event.

_______________ 1. What is the distance between the x- and y-intercepts of the graph of $4x+3y=24$?

$p+q=$ _______________ 2. Let a and b be constants. $x=4$ is the solution to the equation $2x+a=b$. The value for y, such that $3y+b=a$, can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$p+q=$ _______________ 3. The area of the region bounded by the y-axis and lines $y=mx+6$, $y=1$, and $y=4$ is 8. If $m>0$, the value of $m=\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

$m+b=$ _______________ 4. Line ℓ_1 has the equation $2x+3y=24$ and line ℓ_2 has the equation $3x+2y=6$. Line ℓ_3 has the equation $y=mx+b$, where $m>0$. If ℓ_1 is the reflection of ℓ_2 with respect to ℓ_3, determine the value of $m+b$.

Name: ________________________ Team: ________________________

Minnesota State High School Mathematics League

2020-21 Meet 2, Individual Event D

SOLUTIONS

10

1. What is the distance between the x- and y-intercepts of the graph of $4x+3y=24$?

The intercepts are $(6,\ 0)$ *and* $(0,\ 8)$. *These form a 6-8-10 right triangle with the axes.*

−5

2. Let a and b be constants. $x=4$ is the solution to the equation $2x+a=b$. The value for y, such that $3y+b=a$, can be written as $\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

When $x=4$, *the first equation yields* $8=b-a$. *Solving the second equation yields*

$$y=\frac{a-b}{3}\Rightarrow y=\frac{-(8)}{3}=-\frac{8}{3}.$$

37

3. The area of the region bounded by the y-axis and lines $y=mx+6$, $y=1$, and $y=4$ is 8. If $m>0$, the value of $m=\frac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p+q$.

The region is a trapezoid with vertices $(0,\ 1)$, $(0,4)$, $\left(\frac{-2}{m},\ 4\right)$, *and* $\left(\frac{-5}{m},\ 1\right)$. *Therefore, the bases have lengths* $\frac{2}{m}$ *and* $\frac{5}{m}$ *and its height is 3.* $\left(\frac{\frac{2}{m}+\frac{5}{m}}{2}\right)\cdot 3=8\Rightarrow m=\frac{21}{16}$.

19

4. Line ℓ_1 has the equation $2x+3y=24$ and line ℓ_2 has the equation $3x+2y=6$. Line ℓ_3 has the equation $y=mx+b$, where $m>0$. If ℓ_1 is the reflection of ℓ_2 with respect to ℓ_3, determine the value of $m+b$.

Consider the graphs of the lines $2x+3y=0$ *and* $3x+2y=0$. *It is easy to see that the lines* $y=x$ *and* $y=-x$ *are the bisectors of the angles formed by the graphs of these two lines. Lines* ℓ_1 *and* ℓ_2 *have the same slopes as these lines. Similarly, the graph of* ℓ_3 *will bisect the angles formed by* ℓ_1 *and* ℓ_2 *and therefore, will have a slope of 1 (since* $m>0$.*) The intersection of* ℓ_1 *and* ℓ_2 *will also lie on* ℓ_3. *Since* ℓ_1 *and* ℓ_2 *intersect at* $(-6,\ 12)$, *the equation of* ℓ_3 *is*

$$\frac{y-12}{x-(-6)}=1\Rightarrow y=x+18.\ \text{So } m+b=19.$$

Minnesota State High School Mathematics League

2020-21 Meet 2, Team Event

Each question is worth 4 points. Team members may cooperate in any way, but at the end of 30 minutes, submit only one set of answers. Place your answer to each question on the line provided.

______________ 1. In a group of 250 married couples, two-thirds of the husbands who are taller than their wives are also heavier, and three-quarters of the husbands who are heavier than their wives are also taller. There are 30 wives who are both heavier and taller than their husbands. How many husbands are both heavier and taller than their wives, if no husband is equal to his wife in either height or weight?

______________ 2. Three segments, s_1, s_2, and s_3, subsets of the lines represented by the equations $y = 3x$, $y = -5x + 48$, and $y = -\dfrac{x}{2} + 21$, respectively, are three cevians of $\triangle ABC$. Vertex $A = (1,\ 3)$, vertex $B = (40,\ 1)$. How many lattice points in Quadrant I exist for vertex C, so that the point of concurrency of the three cevians lies within $\triangle ABC$?

______________ 3. In $\triangle ABC$, $\tan A = a$ and $\tan B = b$, where a and b are drawn from the set $\{1,\ 2,\ 3\}$, such that a may equal b. For how many ordered pairs $(a,\ b)$ is $m\angle C > 45^\circ$?

$a + b =$ ______________ 4. A triangle has vertices $A(-3,\ 1)$, $B(5,\ 5)$, and $C(12,\ -4)$. A circle, circumscribing this triangle, is centered at $P(a,\ b)$. Determine the value of $a + b$.

$p + q =$ ______________ 5. Line ℓ intersects the line $3x + 2y = -6$ at point A and the line $8x - 15y = -20$ at point B. If $\left(-\dfrac{5}{6},\ -\dfrac{3}{5}\right)$ is the midpoint of $\overline{AB}$, the **sum** of the coordinates of point A can be written as $\dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p + q$.

$AF =$ ______________ 6. Median $\overline{BD}$ of $\triangle ABC$ is extended beyond D to E, so that $DE = \frac{1}{3}BD$. If $EC = 10$, determine the length of median $\overline{AF}$.

Team: ______________________________

Minnesota State High School Mathematics League

2020-21 Meet 2, Team Event

SOLUTIONS (page 1)

120

1. In a group of 250 married couples, two-thirds of the husbands who are taller than their wives are also heavier, and three-quarters of the husbands who are heavier than their wives are also taller. There are 30 wives who are both heavier and taller than their husbands. How many husbands are both heavier and taller than their wives, if no husband is equal to his wife in either height or weight?

5

2. Three segments, s_1, s_2, and s_3, subsets of the lines represented by the equations $y = 3x$, $y = -5x + 48$, and $y = -\dfrac{x}{2} + 21$, respectively, are three cevians of $\triangle ABC$. Vertex $A = (1,\ 3)$, vertex $B = (40,\ 1)$. How many lattice points in Quadrant I exist for vertex C, so that the point of concurrency of the three cevians lies within $\triangle ABC$?

6

3. In $\triangle ABC$, $\tan A = a$ and $\tan B = b$, where a and b are drawn from the set $\{1,\ 2,\ 3\}$, such that a may equal b. For how many ordered pairs $(a,\ b)$ is $m\angle C > 45^\circ$?

1

4. A triangle has vertices $A(-3,\ 1)$, $B(5,\ 5)$, and $C(12,\ -4)$. A circle, circumscribing this triangle, is centered at $P(a,\ b)$. Determine the value of $a + b$.

−5

5. Line ℓ intersects the line $3x + 2y = -6$ at point A and the line $8x - 15y = -20$ at point B. If $\left(-\dfrac{5}{6},\ -\dfrac{3}{5}\right)$ is the midpoint of $\overline{AB}$, the **sum** of the coordinates of point A can be written as $\dfrac{p}{q}$, where p and q are relatively prime integers. Determine the value of $p + q$.

15

6. Median $\overline{BD}$ of $\triangle ABC$ is extended beyond D to E, so that $DE = \frac{1}{3} BD$. If $EC = 10$, determine the length of median $\overline{AF}$.

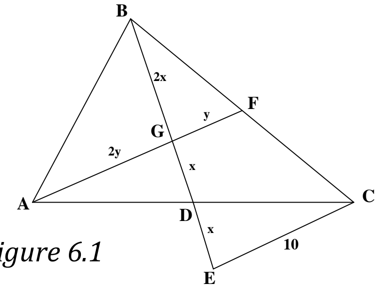


Figure 6.1

Minnesota State High School Mathematics League

2020-21 Meet 2, Team Event

SOLUTIONS (page 2)

1. *Let* $T=$ *number of husbands taller than their wives,* $H=$ *number of husbands heavier than their wives, and* $B=$ *number of husbands both taller and heavier than their wives. Then* $\frac{2}{3}T=B$, $\frac{3}{4}H=B$, *and* $T+H-B+30=250$. *Substituting yields* $\frac{3}{2}B+\frac{4}{3}B-B=220\Rightarrow 9B+8B-6B=1320\Rightarrow B=120$.

2. *The three lines intersect at* $(6,\ 18)$. *Point* A *is on* $y=3x$ *and* B *is on* $y=-\frac{x}{2}+21$, *so* C *must lie on* $y=-5x+48$ *and be between* $(0,\ 48)$ *and* $(6,\ 18)$. *So only five lattice points work for* C: $(1,\ 43)$, $(2,\ 38)$, $(3,\ 33)$, $(4,\ 28)$, *and* $(5,\ 23)$.

3. $\tan C=\tan\left(180-(A+B)\right)=-\tan(A+B)=-\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{\tan A+\tan B}{\tan A\tan B-1}=\frac{a+b}{ab-1}$. *There are nine possible ordered pairs to consider:*

a	1	1	1	2	2	2	3	3	3
b	1	2	3	1	2	3	1	2	3
$\tan C$	$\varnothing$	3	2	3	$\frac{4}{3}$	1	2	1	$\frac{3}{4}$

. *For the* $m\angle C$ *to be greater than* 45°, $\tan C$ *must be greater than* 1, *giving five pairs. But also when* $a=b=1$, $C=90^\circ$. *So a total of six pairs work.*

4. *The midpoint of* $\overline{AB}$ *is* $(1,\ 3)$ *and the slope of* $\overline{AB}$ *is* $\frac{1}{2}$. *So the perpendicular bisector of* $\overline{AB}$ *is* $\frac{y-3}{x-1}=\frac{-2}{1}\Rightarrow 2x+y=5$. *The midpoint of* $\overline{BC}$ *is* $\left(\frac{17}{2},\ \frac{1}{2}\right)$ *and the slope of* $\overline{BC}$ *is* $\frac{-9}{7}$. *So the perpendicular bisector of* $\overline{BC}$ *is* $\frac{y-\frac{1}{2}}{x-\frac{17}{2}}=\frac{7}{9}\Rightarrow 7x-9y=55$. *Since the center of a circle lies on a perpendicular bisector of any chord, the center must be the intersection of these two perpendicular bisectors or* $(4,\ -3)$.

5. *Let* $A=(x_1,\ y_1)$ *and* $B=(x_2,\ y_2)$. *Then* $\frac{x_1+x_2}{2}=-\frac{5}{6}\Rightarrow x_2=-\frac{3x_1+5}{3}$ *and* $\frac{y_1+y_2}{2}=-\frac{3}{5}\Rightarrow y_2=-\frac{5y_1+6}{5}$.
Also $3x_1+2y_1=-6$ *and* $8x_2-15y_2=-20\Rightarrow 8\left(-\frac{3x_1+5}{3}\right)-15\left(-\frac{5y_1+6}{5}\right)=-20\Rightarrow 24x_1-45y_1=74$.
Solving this system of equations yields $y_1=-2$ *and* $x_1=-\frac{2}{3}$. *Then* $-2+\frac{-2}{3}=\frac{-8}{3}$.

6. *In Figure 6.1, medians* $\overline{BD}$ *and* $\overline{AF}$ *intersect at* G, *the centroid, dividing the medians into segments with a ratio of 1:2. Therefore,* $DE=x$ *and* G *is the midpoint of* $\overline{BE}$. *Therefore,* $\overline{GF}$ *is a midsegment of* $\triangle BEC$, *making* $GF=5$ *and* $AF=15$.