Magic Squares

Definitions

A ***magic square*** is a square consisting of n^2 boxes, called ***cells***, filled with integers that are all different. The sums of the numbers in the horizontal rows, vertical columns, and main diagonals are all equal.

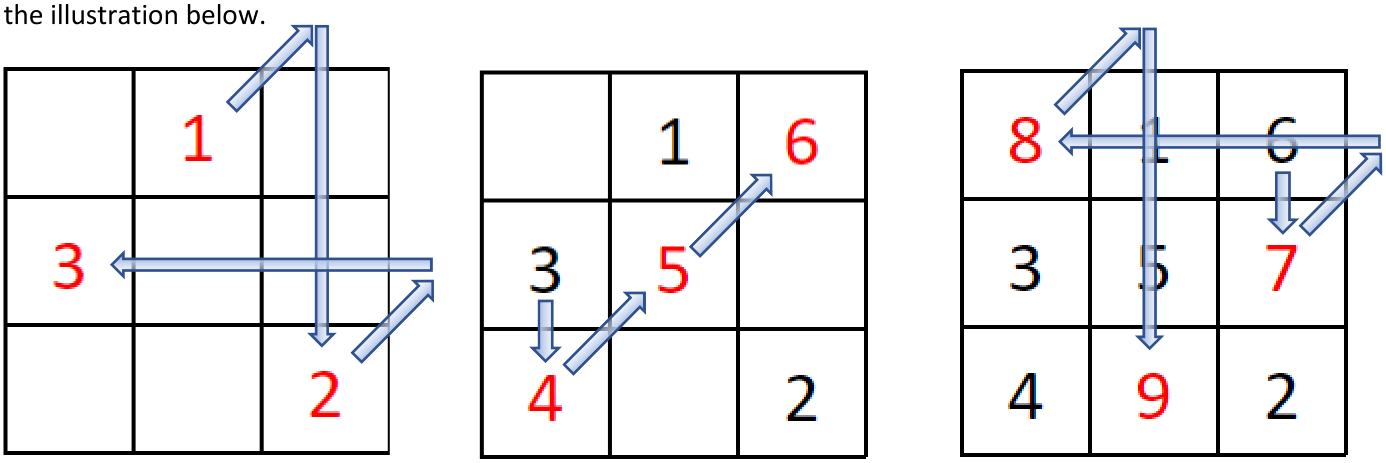
If the integers in a magic square are the consecutive natural numbers from 1 to n^2, the square is said to be of the nth order, and the ***magic number***, or sum of each row, is a constant equal to $n(n^2+1)/2$.

3x3 Magic Square

8	1	6
3	5	7
4	9	2

The 3x3 (or 3rd-order) magic square has a magic number of $\frac{n(n^2+1)}{2} = \frac{(3)[(3)^2+1]}{2} = \frac{3(9+1)}{2} = \frac{3(10)}{2} = \frac{30}{2} = 15$. Each row, column, and main diagonal sum to 15. See the complete 3x3 magic square at right.

One way to construct the 3x3 magic square is as follows. Begin by writing the numeral 1 in the top middle cell. Continue writing the next-higher number in the cell up and to the right. If moving up and to the right takes you above *or* to the right of the magic square, jump to the square that is all the way across the magic square. If moving up and to the right takes you above *and* to the right of the magic square, or to an already filled-in square, drop down a square. See the illustration below.



Another way to construct the 3x3 magic square is to center the numbers on zero, meaning the range of integers is from -4 to 4. In this way, the symmetry is apparent. Place zero in the middle, 1 in the upper-right corner move clockwise, counting up to 4. Negate the 3. The place -1 in the lower-left corner, and continue moving clockwise, counting down to -4. Negate the -3.

3	-4	1
-2	0	2
-1	4	-3

→ +5 →

8	1	6
3	5	7
4	9	2

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4x4 Magic Square

The 4x4 (or 4th-order) magic square has a magic number of $\frac{n(n^2+1)}{2} = \frac{(4)[(4)^2+1]}{2} = 34$. There are 880 distinct forms of this magic square. Perhaps the most famous is ***Dürer's magic square***, given below.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Dürer's magic square appears in an engraving entitled *Melancholia I* by Albrecht Dürer. The engraving shows a disorganized jumble of scientific equipment lying unused while an intellectual sits absorbed in thought. Durer's magic square is in the upper right-hand corner of the engraving. The numbers 15 and 14 appear in the middle of the bottom row, indicating the date of the engraving, 1514.

A slightly different form of the 4x4 magic square lends itself to an easy construction in two steps. In the first step, begin by writing 1 in the upper left-hand corner cell and count up by one in each cell as you move across the row to the right, and continue for each row below, but (and this is the key) *only write in those cells that form an "X" pattern*. See the illustration. In the second step, begin in the lower right-hand corner cell with 16 (already written) and count down by one in each cell as you move across the row to the left, and continue for each row above, only writing in the empty cells.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

=

1			4
	6	7	
	10	11	
13			16

+

	15	14	
12			9
8			5
	3	2	

This is a magic square, so by definition each row, column, and main diagonal sum to the magic number 34. However, there are many other sets of four cells arranged in interesting patterns that also sum to 34. How many can you find? List them as the four numbers in increasing order. For example, write the first row as 1-4-14-15.

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Franklin Magic Square

Benjamin Franklin was a scientist, inventor, statesman, printer, philosopher, musician, writer, economist, and the first U.S. Postmaster General. In 1769, in a letter to a colleague, he describes a magic square he had created early in his life. In fact, Franklin created many magic squares, for fun and as an intellectual challenge.

Franklin's 8x8 (or 8th-order) magic square has a magic number of $\frac{n(n^2+1)}{2} = \frac{(8)[(8)^2+1]}{2} = 260$. It as several unusual properties. Each row and column sum to 260. Each half row or column sums to half of 260. In addition, each of the ***bent rows*** sum to 260. See the gray cells in the illustration for examples of a bent row. See the cells with the thicker black borders for an example of a broken bent row (14+61+64+15+18+33+36+19), which also sums to 260. Numerous other symmetries can be found—for example, the four corner numbers and the four middle numbers sum to 260. The sum of the numbers in any 2x2 subsquare is 130, and the sum of any four numbers that are arranged equidistant from the center of the square also equals 130. When converted to binary numbers, even more startling symmetries are found.

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

Alas, despite all the marvelous symmetries, the main diagonals do not each sum to 260, so this cannot strictly qualify as a magic square according to the common definition. Franklin claimed he could generate the squares "as fast as he could write," though we are not certain as to his method.

How many more interesting combinations of 260 or 130 can you find? (And what about $260 \div 4 = 65$?)

References

Pickover, Clifford A. (2009). *The Math Book: From Pythagoras to the 57th dimension, 250 milestones in the history of mathematics*. Sterling Publishing.

Weisstein, Eric W. "Dürer's Magic Square." From *MathWorld*--A Wolfram Web Resource. https://mathworld.wolfram.com/DuerersMagicSquare.html