Math Team Notes
Topic 1A: Prealgebra

Subtopics

Topic 1A, Prealgebra, includes the following subtopics.

<u>**1A Algebra 1: Prealgebra**</u>
1A1 Fractions to add and express as the quotient of two relatively prime integers
1A2 Complex fractions and continued fractions
1A3 Decimals, repeating decimals
1A4 Percentage, interest, and discount
1A5 Least common multiple, greatest common divisor
1A6 Number bases, change of base

Notes

- **Definition** two integers are ***relatively prime*** if they have no common positive integer factor other than 1. For example, the numbers 8 and 15, though neither is prime ($8 = 2 \times 4$, $15 = 3 \times 5$), have no common factor (other than 1) and are therefore relatively prime. A fraction (such as $\frac{8}{15}$) is said to be ***reduced*** or ***in simplest form*** or ***in lowest terms*** when the numerator and denominator consist of two relatively prime integers.
- **Definition** a ***complex fraction*** is a fraction in which the numerator or the denominator (or both) contains a fraction. For example, the fraction $\frac{1}{2+\frac{3}{4}}$ is a complex fraction, because the denominator $2 + \frac{3}{4}$ contains the fraction $\frac{3}{4}$.
- **Definition** a ***continued fraction*** is a complex fraction that repeats. For example, the fraction $\frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}$ is a continued fraction.
- To evaluate a continued fraction, set x equal to the continued fraction, and substitute x *inside* the fraction for the part that repeats. For example, to evaluate $\frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}$, let $x = \frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}} = \frac{1}{2+x}$, then solve $x = \frac{1}{2+x}$ as $x^2 + 2x - 1 = 0$, or $x = \frac{-2\pm\sqrt{2^2-4(1)(-1)}}{2(1)} = -1 \pm \sqrt{2}$.
- **Definition** a ***repeating decimal*** is a decimal which repeats. For example, the decimal number $0.20\overline{19}$ is a repeating decimal, equal to $0.201919191919\cdots$. The decimal is read, "Zero point two zero repetend one nine," where the horizontal bar above the 19 is the repetend symbol.
- A repeating decimal is a rational number. One way to express a repeating decimal as a rational number is to isolate the repeating portion immediately after the decimal point and then express the repeating portion as a fraction over 9 for one repeating digit, 99 for two repeating digits, 999 for three repeating digits, and so on. For example, $0.\overline{3} = \frac{3}{9} = \frac{1}{3}$ and $0.\overline{19} = \frac{19}{99}$. To express $0.0\overline{3}$, let $x = 0.0\overline{3}$. Then isolate with $10x = 0.\overline{3} = \frac{1}{3}$, so $x = 0.0\overline{3} = \frac{1}{30}$.
- **Definition** the **greatest common divisor** (***GCD* or *gcd***), also called the greatest common denominator, of two positive integers is the largest positive integer that divides evenly (i.e., with no remainder) into each of the two integers. For example, the GCD of 12 and 18 is 6.
- To find the GCD, list the prime factors of each number (in increasing order), identify the common factors (including repeats), then multiply them together. For example, $24 = 2 \cdot 2 \cdot 2 \cdot 3$, $36 = 2 \cdot 2 \cdot 3 \cdot 3$, so $gcd(24,36) = 2 \cdot 2 \cdot 3 = 12$. Note that the GCD can never be greater than the difference of the two numbers.

- **Definition** the ***least common multiple*** (***LCM* or *lcm***) of two positive integers is the smallest positive integer into which each integer divides evenly (i.e., with no remainder). For example, the LCM of 24 and 36 is 72.
- To find the LCM, list the prime factors of each number (in increasing order), identify the maximum number of each prime factor, then multiply. For example, $24 = 2 \cdot 2 \cdot 2 \cdot 3$, $36 = 2 \cdot 2 \cdot 3 \cdot 3$, so $lcm(24,36) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 72$, because 24 has three 2 factors and 36 has two 3 factors.
- We typically work with numbers in base 10. This means that each digit in the number is the product of that digit and the power of the base corresponding to the digit's place in the number. For example, the base-10 number $123 = (1 \times 10^2) + (2 \times 10^1) + (3 \times 10^0) = 100 + 20 + 3 = 123$.
- Numbers not in base 10 can be converted into base 10. The base-4 number 123_4 (the subscript indicates the base) is converted as follows: $123_4 = (1 \times 4^2) + (2 \times 4^1) + (3 \times 4^0) = 16 + 8 + 3 = 27$. If no base is indicated, the number is assumed to be in base 10 (so $27 = 27_{10}$).
- A number in base 10 can be converted into another base. For example, the base-10 number 27 can be changed into base 4 as $27 = (1x16) + (2 \times 4) + (3 \times 1) = (1 \times 4^2) + (2 \times 4^1) + (3 \times 4^0) = 123_4$.
- To change a number from one base to another, neither of which is base 10, it is usually easiest to first change from the initial base to base 10, and then change from base 10 to the final base.
- For bases greater than 10, the extra digits are upper-case Latin letters. For example, the base-16 digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F.
- **Definitions** In financial problems, ***profit*** is defined as ***income*** minus ***expenses***. If the profit is negative, it is called a ***loss***.

Problems
For the following problems, assume a calculator is not allowed unless stated.

<u>Problem #1 ("quickie"; 1 point)</u>
Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. Express $\frac{\frac{4}{3}+\frac{5}{4}}{\frac{3}{4}+\frac{4}{5}}$ as a quotient of two relatively prime integers. (MSHSML 2019-20 1A #1)
2. Express $1 + \frac{\frac{1}{2}}{\frac{1}{3}+\frac{1}{4}}$ as a quotient of two relatively prime integers. (MSHSML 2018-19 1A #1)
3. Express $\frac{\frac{4}{3}-\frac{3}{4}}{\frac{4}{3}+\frac{3}{4}}$ as a quotient of two relatively prime integers. (MSHSML 2017-18 1A #1)
4. Express $\frac{2}{3} + \frac{5}{\frac{5}{3}+\frac{5}{6}}$ as a quotient of two relatively prime integers. (MSHSML 2016-17 1A #1)
5. Determine exactly how many Turkish lira 1 dollar will buy if a hotel room that costs $54 may be obtained for 81 Turkish lira. [calculator allowed] (Based on MSHSML 2015-16 1A #1)
6. If $x = \frac{1}{2}$, $y = \frac{1}{3}$, and $z = \frac{1}{4}$, determine exactly the value of $\frac{x}{y+z}$. (MSHSML 2014-15 1A #1)
7. Determine exactly the value of $\frac{LCM(20,14)}{GCD(20,14)}$. (MSHSML 2013-14 1A #1)

8. The product of the repeating decimals $0.33333\ldots$ and $0.66666\ldots$ is $0.\underline{X}\,\underline{X}\,\underline{X}\,\underline{X}\,\underline{X}\ldots$. What digit does $\underline{X}$ represent? [calculator allowed] (MSHSML 2012-13 1A #1)
9. There were 7 boys and 13 girls at a party. What percentage of the group were boys? (MSHSML 2011-12 1A #1)
10. If $\frac{1}{\left(\frac{1}{\left(\frac{1}{x}\right)}\right)} = \frac{2}{3}$, determine exactly the value of x.

Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. Let b be a positive integer. For how many values of b is 21_b a two-digit number in base 10? (MSHSML 2019-20 1A #2)
2. In May the fish population of Prime Lake was 12100. By June, the population had grown by 2100. However, in July a disease spread through the lake, killing 29% of the fish. After the disease, how many fewer fish were in the lake in July than in May? (MSHSML 2018-19 1A #2)
3. Compute $\frac{lcm(20,18)}{gcd(20,18)}$. (MSHSML 2017-18 1A #2)
4. Find the base-nine number that is equivalent to 245_6. (MSHSML 2016-17 1A #2)
5. Determine the exact value of $\frac{0.\overline{7}}{0.\overline{63}}$. [calculator allowed] (MSHSML 2015-16 1A #2)
6. Express $0.037037037\ldots$ as a fraction $\frac{p}{q}$, where p and q are relatively prime integers. (MSHSML 2014-15 1A #2)
7. The expression $5 - \frac{1}{4-\frac{1}{3-\frac{1}{2-\frac{1}{1}}}}$ can be simplified and expressed as a single rational number. Determine exactly that rational number. (MSHSML 2013-14 1A #2)
8. Write the base-10 number 140 in base 15. [calculator allowed] (MSHSML 2012-13 1A #2)
9. If $161 \equiv x \bmod 13$ and x is a single positive digit, determine the value of x. (MSHSML 2011-12 1A #2)
10. The LCM of 123, 231, and 312 can be written as a power of 2 multiplied by five other prime numbers. Do so. (MSHSML 2010-11 1A #2)

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. Determine exactly the smallest positive rational number which when divided by $\frac{4}{11}$, $\frac{3}{22}$, or $\frac{5}{33}$ always yields an integer. (MSHSML 2019-20 1A #3)
2. Express $0.20\overline{19}$ as a quotient of two relatively prime integers. (MSHSML 2018-19 1A #3)
3. What is the base b for which $\underline{6}\,\underline{8}_{\,b}$ is 25% larger than $\underline{5}\,\underline{3}_{\,b}$? (Note that the percent is given in base 10.) (MSHSML 2017-18 1A #3)
4. If 48 and x have a lowest common multiple of 2640 and a greatest common factor of 12, determine the minimum possible value of x. (MSHSML 2016-17 1A #3)
5. Let a and b be positive integers. If the greatest common factor of a and b is 10 and the least common multiple of a and b is 60, determine the minimum possible value of $a + b$. [calculator allowed] (MSHSML 2015-16 1A #3)
6. Alice and Bob are the only income earners in their family. In 2012, Alice's salary grew by 20%, while Bob's salary dropped by 20%; however, their total family income was unchanged. In 2013, Alice's income again grew by 20%, while Bob's income again dropped by 20%. By what percentage did their family income change in 2013? (MSHSML 2014-15 1A #3)

Express your answer as a percent, including a plus sign (+) to indicate an increase, or a minus sign (−) to indicate a decrease. (MSHSML 2014-15 1A #3)

7. The ePad was originally priced at $\$100$, but the newest model of ePad is coming out, so the old one is going on sale. Each day, the price is reduced by 10%, and rounded down to the nearest dollar, as necessary. (On Day 1, the price is reduced to $\$90$; on Day 2 it is $\$81$; on Day 3 it is $\$72$, and so on.) What is the <u>first</u> day on which the old ePad will cost $\$1$? (MSHSML 2013-14 1A #3)
8. A used-car dealer sold two cars, receiving $560 in payment for each car. One of those transactions resulted in a 40% profit for the dealer, while the other resulted in a 20% loss. What was the dealer's net profit (in dollars) for the two transactions? [calculator allowed] (MSHSML 2012-13 1A #3)
9. Convert $r = 1.4727272\ldots$ into a fraction expressed as the quotient of two relatively prime integers. (MSHSML 2011-12 1A #3)
10. I'm out for lunch at my favorite café, but I only have $15.00. If the soup-and-sandwich combo I want to order costs $13.00, and sales tax is 7%, what is the minimum whole-number percent-off discount coupon I must hold in my wallet to allow me to still leave an 18% tip?
Note: tax and tip are applied after the coupon, but not to each other. (MSHSML 2010-11 1A #3)

<u>Problem #4 ("challenge"; 2 points)</u>

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

1. [To be inserted]
2. [TBI]
3. [TBI]
4. [TBI]
5. [TBI]
6. [TBI]
7. Express $\frac{1}{9} + \frac{1}{11} + \frac{1}{999}$ as a repeating decimal. (MSHSML 2013-14 1A #4)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge in other mathematical areas (geometry, for example). For past MSHSML Meet 1 Event A #4 problems, see previous exams.