Math Team Notes
Topic 1B: Angles and Special Triangles

Subtopics

Topic 1B, Angles and Special Triangles, includes the following subtopics.

<u>1B Geometry: Angles and Special Triangles</u>
1B1 Pythagorean Theorem, common Pythagorean triples
1B2 Complementary, supplementary, and vertical angles
1B3 Interior and exterior angles for triangles and polygons
1B4 Angles formed by transversals cutting parallel lines
1B5 Properties of isosceles and equilateral triangles
1B6 Relationships in $30°$-$60°$-$90°$- and $45°$-$45°$-$90°$- triangles

Notes

- **Pythagorean Theorem** For a right triangle with legs a and b and hypotenuse c, $a^2 + b^2 = c^2$.
- **Converse of the Pythagorean Theorem** If three real numbers satisfy $a^2 + b^2 = c^2$, then a right triangle may be formed with legs a and b and hypotenuse c.
- **Definition** A Pythagorean triple is a set of three positive integers that satisfies $a^2 + b^2 = c^2$. It is typically written as an ordered triple, such as $(3,4,5)$. Other common Pythagorean triples are $(5,12,13)$, $(7,24,25)$, $(8,15,17)$ and $(9,40,41)$. Recognize these and look for them.
- A multiple of a Pythagorean triple is also a Pythagorean triple. For example, multiplying each number of the Pythagorean triple $(3,4,5)$ by 2 yields the Pythagorean triple $(6,8,10)$.
- There are an infinite number of Pythagorean triples. One way to create your own is to choose m and n to be positive integers such that $m > n$. Then $(m^2 - n^2, 2mn, m^2 + n^2)$ is a Pythagorean triple. (Note that because of the variable nature of m and n, the triple may not be ordered.)
- **Definition** Two angles A and B are ***complementary*** if their sum is $90°$.
- **Definition** Two angles A and B are ***supplementary*** if their sum is $180°$.
- **Definition** Two angles A and B are ***vertical angles*** if they form a pair of "opposite" angles formed by two intersecting lines or segments. Vertical angles are congruent (and have equal measure).
- **Definitions** Points are ***collinear*** if there is a line that contains all of them. Points are ***noncollinear*** if no single line contains them all. Lines are ***concurrent*** if they contain the same point.
- **Definition** An ***interior angle*** of a polygon is an angle on the inside of the polygon. The interior angles of a triangle sum to $180°$. For a polygon with n sides (commonly called an ***n-gon***), the interior angles sum to $(n-2)180°$. For example, $180°$ for a triangle ($n = 3$), $360°$ for a rectangle ($n = 4$) and $540°$ for a pentagon ($n = 5$).
- **Definition** An ***exterior angle*** of a polygon is the angle formed between the extension of one side of the polygon and adjoining side. The exterior angles of a triangle sum to $360°$, assuming we only allow one exterior angle per vertex in the "same direction." In fact, the exterior angles of any n-gon sum to $360°$.
- **Definition** A **regular polygon** is a polygon in the sides are all the same length and are symmetrically placed around a common center.
- **Definition** A ***transversal*** is a line that intersects two or more lines at different points.
- The angles formed by a transversal and the two (or more) *parallel* lines it intersects have simple relationships. The names of the angle pairs (e.g., complementary angles, same-side interior angles) are less important than their numerical relationships. Know the numerical relationships.

Math Team Notes
Topic 1B: Angles and Special Triangles

- **Definitions** An angle is ***acute*** iff it is less than $90°$, ***right*** iff it is $90°$, ***obtuse*** iff it is greater than $90°$ but less than $180°$, ***straight*** if it is $180°$.
- **Definition** A triangle is ***scalene*** if it has no equal sides, ***isosceles*** if it has *at least* two equal sides, and ***equilateral*** if all its sides are equal.
- **Definitions** A triangle is ***obtuse*** iff it has an obtuse angle, ***right*** if it has a right angle, **acute** if all its angles are acute, and ***equiangular*** iff all its angles are equal.
- **Theorems** If two sides of a triangle are equal, the angles opposite them are equal. Conversely, if two angles of a triangle are equal, the sides opposite them are equal. An equilateral triangle is equiangular. An equiangular triangle is equilateral.
- **Theorem** In an isosceles right triangle (with angles of $45°$-$45°$-$90°$), the hypotenuse is $\sqrt{2}$ times the length of the leg.
- **Theorem** In a $30°$-$60°$-$90°$ right triangle, the hypotenuse is twice the shorter leg and the longer leg is $\sqrt{3}$ times the shorter leg.
- **Definitions** To ***solve a triangle*** is to determine all its (unknown) side lengths and angle measures. The set of all (known) side lengths and angle measures is the ***solution of the triangle***.

Problems
For the following problems, assume a calculator is not allowed unless stated.

Problem #1 ("quickie"; 1 point)
Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. In *Figure 1*, *ABCDEFGH* is a cube. What is $m\angle EBD$? (Hint: The answer is not $90°$.) [calculator allowed] (MSHSML 2019-20 1B #1)

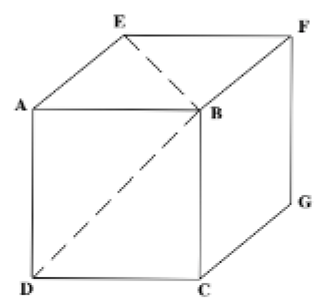


Figure 1

2. Determine exactly the length of the hypotenuse in a right triangle whose legs have lengths of 360 and 480. [calculator allowed] (MSHSML 2018-19 1B #1)
3. Right triangle $\triangle ABC$ has legs of lengths $3\sqrt{2}$ and $4\sqrt{2}$. Determine exactly the length of the hypotenuse. (MSHSML 2017-18 1B #1)
4. A rectangular box has faces whose side lengths are $\sqrt{2}$, 3, and 5. Find the longest diagonal of the box. [calculator allowed] (MSHSML 2016-17 1B #1)
5. In *Figure 1*, if $\triangle ABC$ is equilateral, and $\overline{CD}$ is parallel to $\overline{AB}$, calculate the measure of $\angle BCD$. [calculator allowed] (MSHSML 2015-16 1B #1)

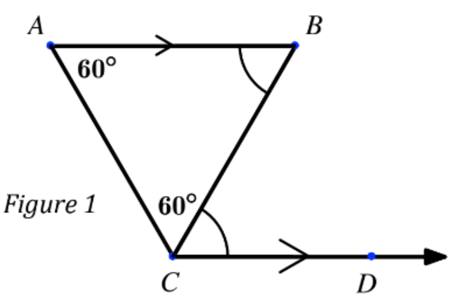


Figure 1

6. In $\triangle ABC$, $m\angle A = 45°$ and $m\angle B = 30°$ as shown in *Figure 1*. If $BC = 12$, determine exactly the length AC. [calculator allowed] (MSHSML 2014-15 1B #1)

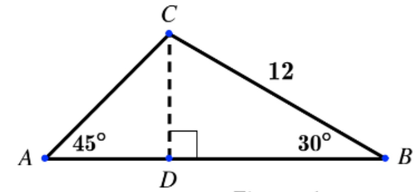
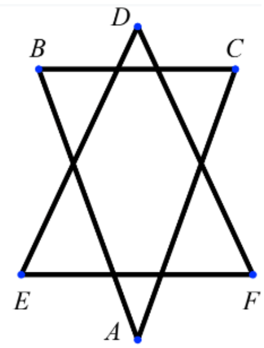


Figure 1

7. If $\frac{1}{3}$ and $\frac{1}{4}$ are the lengths of the two legs of a right triangle, determine exactly the length of the triangle's hypotenuse. [calculator allowed] (MSHSML 2013-14 1B #1)
8. *Figure 1* shows a star-like object formed by overlaying two isosceles triangles: $\triangle ABC$ with apex angle $A = 40°$ and $\triangle DEF$ with apex angle $D = 50°$. Calculate $m\angle A + m\angle B + m\angle C + m\angle D + m\angle E + m\angle F$. [calculator allowed] (MSHSML 2012-13 1B #1)

Figure 1

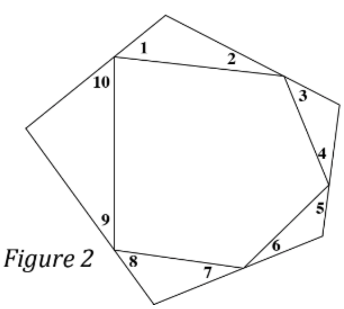


Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 ("textbook") in less than two minutes.

1. In *Figure 2*, determine exactly the sum of the angles labeled 1 through 10. [calculator allowed] (MSHSML 2019-20 1B #2)

Figure 2



2. Three non-concurrent lines are drawn in a plane. Lines l_1 and l_2 intersect in an acute angle of $50°$ and lines l_2 and l_3 at an acute angle of $20°$. Determine exactly all possible values (in degrees) for the measure of the <u>acute</u> angle at which l_1 and l_3 meet. [calculator allowed] (MSHSML 2018-19 1B #2)
3. Equilateral $\triangle ABC$ has side length of 5. Point D is in the interior of $\triangle ABC$ such that $\triangle DCB$ is an isosceles right triangle. Determine exactly AD. (MSHSML 2017-18 1B #2)
4. $\triangle ABC$ is an isosceles triangle whose hypotenuse $\overline{AC}$ has a length of $9\sqrt{6}$. If point D lies on $\overline{BC}$ such that $m\angle BAD = 30°$, determine exactly AD. [calculator allowed] (MSHSML 2016-17 1B #2)
5. Town A is located exactly 120 miles north of town B. If Sue hops in a car and drives directly east from town B at 50 mph, calculate how many hours (as a decimal) it will take for Sue to be exactly 241 miles from town A as the crow flies. [calculator allowed] (MSHSML 2015-16 1B #2)

6. In *Figure 2*, lines l_1 and l_2 are parallel, while lines l_3 and l_4 intersect at an angle of $17°$. If the acute angle formed by l_1 and l_4 measures $44°$, calculate the measure of the <u>obtuse</u> angle between l_1 and l_3. [calculator allowed] (MSHSML 2014-15 1B #2)

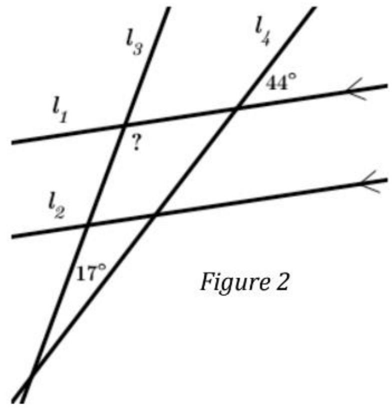


Figure 2

7. In *Figure 2*, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overleftrightarrow{PQ}$ is a transversal, $m\angle BPQ = 2x$, and $m\angle DQP = 5x - 51$. Determine exactly the measure of angle ADQ. [calculator allowed] (MSHSML 2013-14 1B #2)

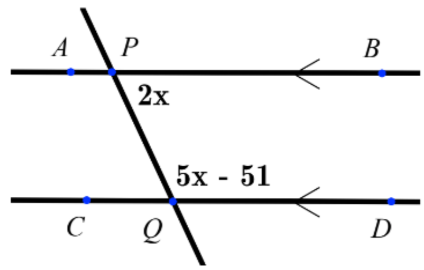


Figure 2

8. Beginning with regular pentagon $A_1A_2A_3A_4A_5$, we form five-pointed star $S_1S_2S_3S_4S_5$ by extending the edges of the pentagon, as shown in *Figure 2*. Calculate $m\angle S_1 + m\angle S_2 + m\angle S_3 + m\angle S_4 + m\angle S_5$, where S_1, S_2, S_3, S_4, S_5 are the <u>acute</u> angles at each vertex. [calculator allowed] (MSHSML 2012-13 1B #2)

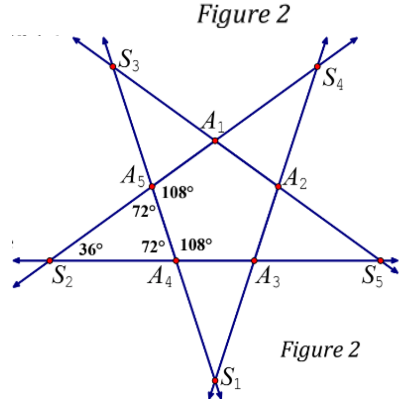


Figure 2

<u>Problem #3 ("textbook with a twist"; 2 points)</u>

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. The interior angles of a convex polygon increase in the following linear progression: $100°$, $108°$, $116°$, …. Determine the number of sides of the polygon. [calculator allowed] (MSHSML 2019-20 1B #3)
2. Determine exactly both real numbers x such that $\sqrt{x+2}$, $\sqrt{3x-2}$, and $\sqrt{6x-5}$ are the side lengths of a right triangle. Express answers as quotients of relatively prime integers. [calculator allowed] (MSHSML 2018-19 1B #3)

3. In *Figure 3*, $\triangle ABC$ is an isosceles right triangle with hypotenuse $\overline{AC}$. $\overline{BD} \perp \overline{AF}$, $\overline{DE} \perp \overline{BC}$, and $m\angle ABD = 60°$. If $AF = 5\sqrt{6}$, determine exactly the length of $\overline{CE}$. (MSHSML 2017-18 1B #3)

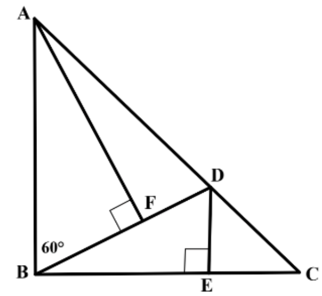


Figure 3

4. Given $m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 = y$, as shown in *Figure 3*. Find the smallest possible angle y (in degrees) if x is an obtuse angle with an integer measure. [calculator allowed] (MSHSML 2016-17 1B #3)

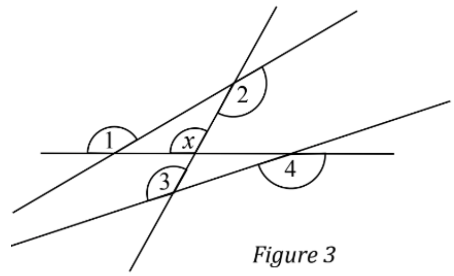


Figure 3

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge in other mathematical areas (algebra, for example). For past MSHSML Meet 1 Event B #4 problems, see previous exams.