Math Team Notes
Topic 1C: Elementary Trigonometry

Subtopics

Topic 1C, Elementary Trigonometry, includes the following subtopics.

<u>**1C Precalculus & Trigonometry: Elementary Trigonometry**</u>
1C1 Definitions and solution of right triangles
1C2 Elementary identities
1C3 Radian measure and graphs of elementary functions
1C4 Trigonometric functions of multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$.

Notes

- **Definitions** A triangle is ***obtuse*** iff it has an obtuse angle, ***right*** if it has a right angle, **acute** if all its angles are acute, and ***equiangular*** iff all its angles are equal.
- **Theorem** In an isosceles right triangle (with angles of $45°$-$45°$-$90°$), the length of the hypotenuse is $\sqrt{2}$ times the length of the leg.
- If the hypotenuse of an isosceles right ($45°$-$45°$-$90°$) triangle has length 1, each leg has length $\frac{\sqrt{2}}{2}$.
- **Theorem** In a $30°$-$60°$-$90°$ right triangle, the length of the hypotenuse is twice the length of the shorter leg and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.
- If the hypotenuse of a $30°$-$60°$-$90°$ right triangle has length 1, the short leg has length $\frac{1}{2}$, and the long leg has length $\frac{\sqrt{3}}{2}$.
- **Definitions** The ***sine*** of an acute angle of a right triangle is the ratio of the length of the opposite leg to the length of the hypotenuse. The ***cosine*** of an acute angle of a right triangle is the ratio of the length of the adjacent leg to the length of the hypotenuse. The ***tangent*** of an acute angle of a right triangle is the ratio of the length of the opposite leg to the length of the adjacent leg.
- **Mnemonic** The sine, cosine, and tangent definitions may be remembered by use of the acronym SOH-CAH-TOA (sine is opposite over hypotenuse, cosine is adjacent over hypotenuse, tangent is opposite over adjacent). Some prefer the more visual $\mathrm{S}\frac{\mathrm{O}}{\mathrm{H}}\mathrm{C}\frac{\mathrm{A}}{\mathrm{H}}\mathrm{T}\frac{\mathrm{O}}{\mathrm{A}}$.
- **Definitions** To ***solve a triangle*** is to determine all its (unknown) side lengths and angle measures. The set of all (known) side lengths and angle measures is the ***solution of the triangle***.
- **Definition** The ***tangent*** of an angle (not necessarily acute) is the ratio of the sine of the angle to the cosine of the angle: $\tan A = \frac{\sin A}{\cos A}$.
- **Definitions** The ***cosecant***, ***secant***, and ***cotangent*** are the reciprocals of the sine, cosine, and tangent, respectively.

$\sin A = \frac{\text{opp}}{\text{hyp}}$	$\csc A = \frac{1}{\sin A}$
$\cos A = \frac{\text{adj}}{\text{hyp}}$	$\sec A = \frac{1}{\cos A}$
$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{\sin A}{\cos A}$	$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$

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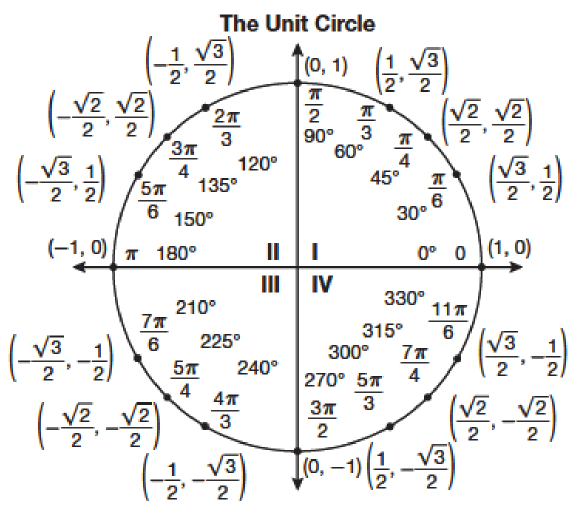
- **Definitions** A circle has $360°$ and 2π ***radians*** (abbreviated ***rad***). An angle with no units is assumed to be in radians.[1] Thus $360° = 2\pi$, so, as strange as it may look, $\frac{2\pi}{360°} = \frac{360°}{2\pi} = 1$ (see note[2]). Use the factors $\frac{2\pi}{360°} = \frac{\pi}{180°}$ and $\frac{360°}{2\pi} = \frac{180°}{\pi}$ as necessary to convert from degrees to radians or vice-versa. For example, to find $72°$ in radians, multiply by $\frac{\pi}{180°}$ (chosen so that the $180°$ in the denominator will "cancel" the degrees in $72°$) to obtain $(72°)\frac{\pi}{180°} = \frac{2\pi}{5}$.
- **Theorem** The sine, cosine, and tangent of angles $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$ are as follows. These need not be memorized (though it helps); rather, sketch a $45°$-$45°$-$90°$ or $30°$-$60°$-$90°$ right triangle, as appropriate, with a hypotenuse length of 1 and apply $\mathrm{S}\frac{\mathrm{O}}{\mathrm{H}}\mathrm{C}\frac{\mathrm{A}}{\mathrm{H}}\mathrm{T}\frac{\mathrm{O}}{\mathrm{A}}$. Be sure to rationalize the denominator if necessary.

	Angle A			
	$\frac{\pi}{6}$ rad $= 30°$	$\frac{\pi}{4}$ rad $= 45°$	$\frac{\pi}{3}$ rad $= 60°$	$\frac{\pi}{2}$ rad $= 90°$
$\sin A$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos A$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan A$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	[undefined]

- Use the ***unit circle***, a circle of radius 1 centered at the origin, to determine the sine and cosine of any angle that is a multiple of $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$ as the x-coordinate and y-coordinate, respectively, of the point on the circle intersected by the ray with endpoint at the origin that makes the given angle with the positive x-axis. By convention, a positive angle is measured counter-clockwise from the x-axis (toward the positive y-axis) and a negative angle is measured in the opposite direction (i.e., clockwise from the x-axis, away from the positive y-axis). Use unit circle symmetry to determine sine and cosine values of angles in quadrants II, III, and IV, by relating to the known sine and cosine values in quadrant I.



- **Identities**

$\sin^2\theta + \cos^2\theta = 1 \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

$\sin 2\theta = 2\sin\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$

[1] Sometimes problem writers forget to write the degree symbol. For example, a figure of a triangle may have an angle marked "45," with no degree symbol. If it seems obvious, assume degrees. If you are unsure, either (1) ask (if allowed) or (2) state your assumption, solve the problem, and (if necessary) submit a challenge afterward.
[2] It is the presence of the units of degrees (°) and implied units of radians that makes this statement true. After all, even though we know $1 \neq 100$, we understand that 1 dollar = 100 pennies.

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- **Theorem** If angles x and y are in Quadrant 1 and $\sin x = \cos y$, then $x + y = 90°$.
- **Definitions** The graph of $y = a\sin[b(x-c)] + d$ has ***amplitude*** $|a|$, ***period*** $\frac{2\pi}{b}$, ***horizontal shift*** (or ***phase shift***, "delayed" to the *right* because of the minus sign) c, and ***vertical shift*** d. The ***frequency*** is number of periods in the interval 0 to 2π and is equal to b. Note the identities $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$ and $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$ follow from the phase shift definition.

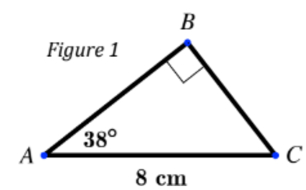
Problems

For the following problems, assume a calculator is not allowed unless stated.

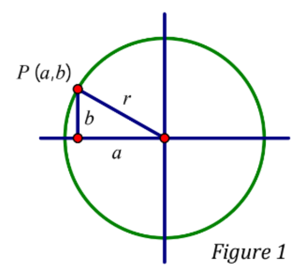
Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

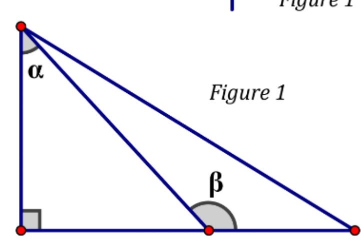
1. Determine exactly the value of $\sin\frac{\pi}{3} + \tan\frac{\pi}{4} + \cos\frac{\pi}{6}$. (MSHSML 2019-20 1C #1)
2. Determine exactly the value of $\tan\frac{\pi}{6} + \cot\frac{\pi}{3}$. (MSHSML 2018-19 1C #1)
3. In ΔABC, if $\cos A = -\frac{1}{\sqrt{3}}$, determine exactly the value of $\sin A$. (MSHSML 2017-18 1C #1)
4. Determine exactly the value of $\sin\theta + \cos\theta$ if $\theta = \frac{5\pi}{4}$. (MSHSML 2016-17 1C #1)
5. Determine exactly the value of $\sin\frac{\pi}{3} - \cos 3\pi$. (MSHSML 2015-16 1C #1)
6. *Figure 1* shows ΔABC with $m\angle A = 38°$ and $AC = 8$ cm. Calculate the length of $\overline{AB}$. [calculator allowed] (MSHSML 2014-15 1C #1)



7. In *Figure 1*, point $P(a, b)$ is located in the second quadrant on the circumference of a circle of radius r. Express a in terms of b and r. [calculator allowed] (MSHSML 2013-14 1C #1)



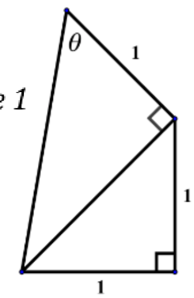
8. In the right triangle shown in *Figure 1*, $\tan\alpha = 1$. Determine exactly the value of $\sin\beta$. (MSHSML 2012-13 1C #1)



9. Determine exactly the value of $\cos\frac{\pi}{1} + \cos\frac{\pi}{2} + \cos\frac{\pi}{3}$. (MSHSML 2011-12 1C #1)

10. Using *Figure 1*, showing certain segments labeled as length 1, determine exactly the value of $\cos\theta$.

Figure 1



Problem #2 ("textbook"; 2 points)

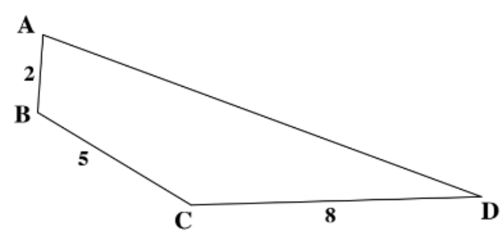
Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. Determine exactly the smallest positive integer n such that $\sec(400°)\cdot\sin(n°) = 1$. (MSHSML 2019-20 1C #2)
2. $\cot\theta = \frac{a^2-b^2}{2ab}$ when $a > b > 0$ and $0° < \theta < 90°$. Write an expression for $\csc\theta$ in terms of a and b. (MSHSML 2018-19 1C #2)
3. For x in radians, $\frac{\pi}{2} < x < \frac{3\pi}{2}$, if $\cot x = 3$, determine exactly the value of $\sec^2 x \cdot \csc x$. (MSHSML 2017-18 1C #2)
4. If $\sin x = \frac{1}{3}$ and $0 < x < \frac{\pi}{2}$, determine exactly the value of $\cos x$. (MSHSML 2016-17 1C #2)
5. If $\tan A = -\frac{\sqrt{39}}{5}$ and $\cos A = \frac{5}{8}$, determine exactly the value of $1 + \sin^2 A$. (MSHSML 2015-16 1C #2)
6. If $\sin x = -\frac{1}{3}$ and $\pi < x < \frac{3\pi}{2}$, determine exactly the value of $\tan x$. [calculator allowed] (MSHSML 2014-15 1C #2)
7. A line segment drawn from the origin to a point (m, n) in the fourth quadrant makes an angle of θ with the positive y-axis. Express θ in terms of m and n. [calculator allowed] (MSHSML 2013-14 1C #2)
8. If $\sin\phi = \frac{5}{13}$, determine exactly all possible values of $\tan\phi$. (MSHSML 2012-13 1C #2)
9. Find the radian measure of the smallest positive angle A for which $\cos A$ has the same value as $\sin 210°$. (MSHSML 2011-12 1C #2)
10. Given that the ratio of $\cos x$ to $\sin x$ is 3: 2, determine exactly the ratio $(\tan x)$:$(\cot x)$. (MSHSML 2010-11 1C #2)

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. $\triangle ABC$ has a right angle at B. If $BC = 1$ and $\cos A = \frac{1}{3}$, determine exactly the perimeter of the triangle. (MSHSML 2019-20 1C #3)
2. In the figure, $AB = 2$, $BC = 5$, and $CD = 8$. Angles A and D are acute and angles B and C are obtuse. If $\sin C = \frac{3}{5}$ and $\cos B = -\frac{3}{5}$, determine exactly AD. (MSHSML 2018-19 1C #3)



3. Determine exactly the value of $\sin 30° + \sin 60° + \sin 90° + \cdots + \sin 300°$. (MSHSML 2017-18 1C #3)
4. If $\sin^2 A = \frac{9}{16}$ and A is in the second quadrant, determine exactly the value of $\tan A$. (MSHSML 2016-17 1C #4)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge in other mathematical areas (algebra or geometry, for example). For past MSHSML Meet 1 Event C #4 problems, see previous exams.