Math Team Notes
Topic 1D: Roots of Quadratic and Polynomial Equations

Subtopics
Topic 1D, Roots of Quadratic and Polynomial Equations, includes the following subtopics.

<u>**1D Algebra 2 & Analysis: Roots of Quadratic and Polynomial Equations**</u>
1D1 Solution of quadratic equations by factoring, by completing the square, by formula
1D2 Complex roots of quadratic equations; the discriminant and the character of the roots
1D3 Relations between roots and coefficients
1D4 Synthetic division
1D5 Function notation

Notes

- **Definition** Given a quadratic of the form $x^2 + bx$, add to it the square of half the coefficient of x, $\left(\frac{b}{2}\right)^2$, to create a perfect square trinomial: $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$. This process is called ***completing the square***.
- **Theorem (Quadratic Formula)** The solutions of the quadratic equation $ax^2 + bx + c = 0$ (with $a \neq 0$), are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Do not forget the presence of a in the denominator.
- Know how to factor a (factorable) quadratic expression $ax^2 + bx + c$ for which $a \neq 1$. Know how to determine that an unfactorable quadratic expressions is, in fact, unfactorable.
- **Definition** The ***discriminant*** is the expression $b^2 - 4ac$ (inside the radical) in the quadratic formula.
- **Theorem** For $ax^2 + bx + c = 0$ and the related function $y = ax^2 + bx + c$,
  - When $b^2 - 4ac > 0$, the equation has 2 real roots, and the graph of the function has 2 x-intercepts
  - When $b^2 - 4ac = 0$, the equation has 1 real root with multiplicity 2, and the graph of the function has 1 x-intercept (it "kisses" the x-axis)
  - When $b^2 - 4ac < 0$, the equation has 0 real roots and 2 complex roots, and the function has 0 x-intercepts
- **Definitions** ***Polynomial roots*** (usually just ***roots***) are the solutions of a ***polynomial equation***. These roots are the ***zeros*** of the related polynomial function, which are the x-intercepts of the graph of the function. For example, the roots of the equation $x^2 - 3x + 1 = -1$ are the zeros of the related polynomial function $f(x) = x^2 - 3x + 2$. (The function is obtained by collecting all nonzero terms on one side of the equation.) The zeros are found by solving $f(x) = 0$, resulting in $x = 1$ and $x = 2$. Hence, the roots of the polynomial equation $x^2 - 3x + 1 = -1$ are $x = 1,2$; the zeros of $f(x) = x^2 - 3x + 2$ are $x = 1,2$; and the x-intercepts of the graph of $f(x)$ are $x = 1,2$, or, alternately, $(1,0)$ and $(2,0)$.
- Know how to use a graphing calculator to determine the approximate x-intercepts of a graph.
- **Definition** To divide polynomial $P(x) = ax^n + bx^{n-1} + \cdots$ by $x - k$, perform ***synthetic division*** as follows:
  - Write k (not $-k$) and the coefficients of $P(x)$ on the first line. Separate the k from the coefficients in some way. Remember to insert any missing zeros.
  - Leave a blank second line. Bring down a to the third line; this is the first coefficient of the quotient.

  - Multiply this quotient coefficient by k and write the result, ka, beneath the next $P(x)$ coefficient, b. Add b to ka. The result, $b + ka$, written in the third line, is the second quotient coefficient.
  - Continue the pattern. Multiply each new quotient coefficient by k and add the result to the next coefficient of $P(x)$. The result of the last addition is the remainder r.
- **Remainder Theorem** If a polynomial $P(x)$ is divided by $x - k$, the remainder is $r = P(k)$.
- **Definition** By the Remainder Theorem, you can divide $P(x)$ by $x - k$ to find $P(k)$. If synthetic division is used to divide, the process is called ***synthetic substitution***. Example: Use synthetic substitution to find $P(5)$ for $P(x) = -2x^4 + 6x^3 + 15x - 1$. The correct answer is $P(x) = -426$.
- **Factor Theorem** For polynomial $P(x)$, $(x - a)$ is a factor of $P(x)$ if and only if $P(x) = 0$.
- If a polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ has rational roots, use the **Rational Root Theorem** to find these roots. To do so, follow these steps:
  - Find all the factors of a_n and a_0.
  - Let q be one of the factors of a_n and p be one of the factors of a_0. List all possible rational numbers $\frac{p}{q}$.
  - Test if $\left(x - \frac{p}{q}\right)$ is a factor of the polynomial using synthetic division. If a polynomial has a rational root, then it will be one of the $\frac{p}{q}$ terms.
- **Vieta's Theorem (for quadratic equations)** For the quadratic equation $ax^2 + bx + c = 0$ with roots r_1 and r_2, the following are true: $r_1 + r_2 = -\frac{b}{a}$ and $r_1 \cdot r_2 = \frac{c}{a}$.
- **Vieta's Theorem (for sum of roots)** For a general polynomial of degree n, $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, the sum of the n roots is $r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}$. For example, for the polynomial $7x^3 - 6x^2 + 5x - 4$, the sum of the $n = 3$ roots is $r_1 + r_2 + r_3 = -\frac{a_{3-1}}{a_3} = -\frac{a_2}{a_3} - \frac{(-6)}{7} = \frac{6}{7}$.
- **Vieta's Theorem (for product of roots)** For a general polynomial of degree n, $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, the product of the n roots is $r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}$
- **Vieta's Theorem (for sum of products of pairs)** For a general polynomial of degree n, $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, the sum of the products of all paired roots is $(r_1 r_2 + r_1 r_3 + \cdots + r_1 r_n) + (r_2 r_3 + r_2 r_4 + \cdots + r_2 r_n) + \cdots + r_{n-1} r_n = \frac{a_{n-2}}{a_n}$
- **Notation** The inverse of a function $f(x)$ is written as $f^{-1}(x)$. Generally[1], the inverse function is an "undo" of the function, so $f\left(f^{-1}(x)\right) = f^{-1}(f(x)) = x$.

Problems

For the following problems, assume a calculator is not allowed unless stated. Note that, even when a calculator is allowed, it may not be *necessary* or even helpful (a calculator may have been allowed on the exam in order to solve a different problem).

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[1] The caveat recognizes that the result assumes the function is a *bijection*, whereby the domain of the inverse is the same as the range of the function and the range of the inverse is the same as the domain of the function.

Math Team Notes
Topic 1D: Roots of Quadratic and Polynomial Equations

Problem #1 ("quickie"; 1 point)
Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. Given $f(x) = 3x^5 + 5x^3 - 2x^2 + 82$, determine exactly $f\left(f^{-1}\left(f(1)\right)\right)$. [calculator allowed] (MSHSML 2019-20 1D #1)
2. Determine exactly all real solutions to the equation $x^2 + 8x = 8$. (MSHSML 2018-19 1D #1)
3. Determine exactly the remainder when $x^3 - 6x^2 + 4x - 5$ is divided by $x - 3$. (MSHSML 2017-18 1D #1)
4. Determine exactly the product of the zeros of the equation $(2x - 7)^2 = 36$. (MSHSML 2016-17 1D #1)
5. Let $f(x) = x + 3$ and $g(x) = x^2$. Determine exactly the value(s) of x for which $g\left(f(x)\right) = 0$. (MSHSML 2015-16 1D #1)
6. Determine exactly the sum of the roots of the cubic polynomial $2x^3 - 9x^2 + 14x - 6$. (MSHSML 2014-15 1D #1)
7. Let r_1 and r_2 be the distinct roots of $r^2 - r - 20$, with $r_1 < r_2$. Determine r_2 exactly. (MSHSML 2013-14 1D #1)
8. Express $(x + 1)(x + 10) + (x + 4)(x - 4)$ as the product of two binomials, each with integer coefficients. [calculator allowed] (MSHSML 2012-13 1D #1)
9. Write, in $x^2 + bx + c = 0$ form, the quadratic equation whose roots are $x = -3$ and $x = 1$. [calculator allowed] (MSHSML 2011-12 1D #1)
10. Determine exactly the least value of x that satisfies the equation $(x - 4)(x + 4) = 9$. [calculator allowed] (MSHSML 2010-11 1D #1)

Problem #2 ("textbook"; 2 points)
Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. $f(x) = x^2 + bx + 12$. Determine for how many integer values of b, $f(x)$ has non-real zeros. [calculator allowed] (MSHSML 2019-20 1D #2)
2. The solutions to $2x^2 + bx + c = 0$ are b and c, where neither is zero. Determine exactly the ordered pair (b, c). (MSHSML 2018-19 1D #2)
3. For what values of m does the product of the roots of $4(x - 2m)^2$ equal 11? (MSHSML 2017-18 1D #2)
4. For what value of a does the polynomial $3x^2 + ax + 10$ have 2 as a root? (MSHSML 2016-17 1D #2)
5. Find the remainder when $2x^3 - 9x^2 + 14x - 6$ is divided by $x + 2$. (MSHSML 2015-16 1D #2)
6. Determine exactly the value of k for which the two solutions of $3x^2 - 4x + k = 0$ are equal. (MSHSML 2014-15 1D #2)
7. Let x_1 and x_2 be the solutions of $x^2 - 20x + 13 = 0$. Determine $\frac{1}{x_1} + \frac{1}{x_2}$ exactly. (MSHSML 2013-14 1D #2)
8. What is the **greatest** integer c for which the quadratic polynomial $5x^2 + 11x + c$ has two distinct rational roots? [calculator allowed] (MSHSML 2012-13 1D #2)
9. Find the remainder when $x^{13} + 1$ is divided by $x - 1$. [calculator allowed] (MSHSML 2011-12 1D #2)
10. Determine exactly the coordinates (both of them) of the highest point of the graph of $y + x^2 + 6x = 4$. [calculator allowed] (MSHSML 2010-11 1D #2)

Problem #3 ("textbook with a twist"; 2 points)
Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. $f(x) = ax^2$ with $a > 0$. An equilateral triangle with side length k is placed on the parabola so that one of its vertices is on the vertex of the parabola and the other two vertices are on $f(x)$.

Write a formula for a, the leading coefficient of $f(x)$, in terms of k. (Be sure to simplify). [calculator allowed] (MSHSML 2019-20 1D #3)

2. The function $f(x) = x^3 + bx^2 + cx + 52$ has $\frac{13}{2-3i}$ as one of its zeros. Determine exactly the ordered pair (b, c). (MSHSML 2018-19 1D #3)
3. For what values of p will the quadratic function $f(x) = x^2 - 4px - 9$ have a minimum value of -333? (MSHSML 2017-18 1D #3)
4. Determine exactly all values of k for which the polynomials $x^2 + 2x - 5k$ and $x^2 - 10x - k$ share a common zero. (MSHSML 2016-17 1D #3)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge in other mathematical areas (geometry, for example). For past MSHSML Meet 1 Event D #4 problems, see previous exams.