Math Team Notes
Topic 2B: Triangular Figures and Solids

Subtopics
Topic 2B, Triangular figures and solids, includes the following subtopics.

2B Geometry: Triangular figures and solids
2B1 Medians, angle bisectors, and altitudes
2B2 Ceva's Theorem and Stewart's Theorem
2B3 Area of a triangle (including Hero's Formula)
2B4 Triangular prisms and pyramids (including surface area and volume)

Notes
- **Definition** A point is a ***midpoint of a line segment*** iff it divides the line segment into two equal segments.
- **Definition** A ***median*** of a triangle is a line segment that joins a vertex to the midpoint of the opposite side. A triangle has three medians, each of which is interior to the triangle (except for its endpoints which are on the triangle).
- **Theorem** In $\triangle ABC$, with side lengths a, b, and c opposite vertices A, B, and C, respectively, the median from C to $\overline{AB}$ (i.e., to the side with length c) has length $\frac{\sqrt{2a^2+2b^2-c^2}}{2}$.
- **Definition** Two or more lines are ***concurrent*** if they contain the same point.
- **Theorem** The medians of a triangle are concurrent.
- **Definition** The ***centroid*** of a triangle is the point in which its medians are concurrent.
- **Definition** A line ***bisects an angle*** iff it divides the angle into two equal angles.
- **Definition** A circle is ***inscribed in a polygon*** iff each side of the polygon is tangent to the circle. The polygon is said to be ***circumscribed about the circle***. The circle is called the ***incircle*** of the polygon, and its center is called the ***incenter*** of the polygon.
- **Theorem** Every triangle has an incircle.
    - **Corollary** The angle bisectors of a triangle are concurrent.
    - **Definition** A ***corollary*** is a theorem that can be easily proved as a consequence of a postulate or another theorem.
- **Theorem** The angle bisector of an angle in a triangle divides the opposite side in the same ratio as the sides adjacent to the angle.
- **Definition** The ***distance between a point and a line*** is the length of the perpendicular segment from the point to the line.
- **Definition** An ***altitude*** of a triangle is the distance from a vertex to the line containing the opposite side. A triangle has three altitudes. Note that in an obtuse triangle (one with an angle greater than 90°), two altitude segments are exterior to the triangle.
- **Theorem** The area of a right triangle is half the product of its legs. In other words, for right triangle ΔABC with right angle C (and thus hypotenuse c, and legs a and b), the area is $Area = \frac{1}{2}ab$.
- **Theorem** The area of a triangle is half the product of any base and corresponding altitude. (This is a generalization of the formula $Area = \frac{1}{2}ab$ to any triangle. You can think of b as any base (or side), and a being its corresponding altitude.)
    - **Corollary** Triangles with equal bases and equal altitudes have equal areas.
- **Heron's Theorem** (aka **Hero's Theorem** or **Hero's Formula**) The area of a triangle with sides a, b, and c is $Area = \sqrt{s(s-a)(s-b)(s-c)}$, where s is the *semiperimeter*, equal to half of the triangle's perimeter, or $s = \frac{1}{2}(a+b+c)$.
- **Theorem** The lines containing the altitudes of a triangle are concurrent.

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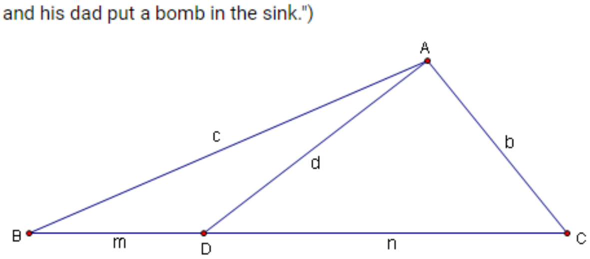
- **Definition** The ***orthocenter*** of a triangle is the point in which the lines containing its altitudes are concurrent.
- **Definition** A ***cevian*** (CHAY-vee-un) of a triangle is a line segment that joins a vertex of the triangle to a point on the opposite side. Note that a median is a cevian, but a cevian is not necessarily a median.
- **Ceva's Theorem** (CHAY-vuhz) Three cevians $\overline{AY}$, $\overline{BZ}$, and $\overline{CX}$ of ΔABC are concurrent iff $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = 1$.
- **Definition** A ***prism*** is a three-dimensional solid object consisting of two ***bases*** (congruent polygons and their interiors that lie in parallel planes) and all the segments that connect a point in one base to a point in the other base. The parallel line segments that connect the corresponding vertices of these bases help form the n ***lateral faces***, where n is the number of sides of each base. For example, a ***triangular prism*** has 2 congruent triangular bases and $n = 3$ lateral (side) faces.
- **Definition** The ***lateral area*** of a prism is the sum of the areas of its lateral faces.
- **Definition** The ***total area*** or ***surface area*** of a prism is the sum of its lateral area and the area of its bases.
- **Definition** The ***altitude of a prism*** is a line segment that connects the planes of its bases and that is perpendicular to both of them.
- **Postulate** The ***volume of a prism*** is the product of the area of its base and its altitude: $V = Bh$.
    - **Corollary** The volume of a rectangular solid is the product of its length, width, and height: $V = lwh$.
    - **Corollary** The volume of a cube is the cube of its edge: $V = e^3$.
    - **Corollary** The volume of a triangular prism is $V = Bh = \frac{1}{2}abh$, where a is the altitude of the base corresponding to base side b, and h is the altitude of the prism.
- **Definitions** A ***pyramid*** is a three-dimensional solid object consisting of one ***base*** (a polygon and its interior that lies in one plane), a point P not in the same plane as the base, and all the segments that connect P to a point in the base. The faces that are not the base are the ***lateral faces*** and the edges in which they intersect each other are the ***lateral edges***. The lateral edges meet at the ***apex*** of the pyramid.
- **Definition** the ***altitude of a pyramid*** is the perpendicular line segment connecting the apex to the plane of its base. It is also the length of this segment.
- **Theorem** The volume of any pyramid is one-third of the product of the area of its base and its altitude: $V = \frac{1}{3}Bh$.
- **Stewart's Theorem** (From Art of Problem Solving). Note that, in the figure, $a = m + n$.

Given a triangle $\triangle ABC$ with sides of length a, b, c opposite vertices are A, B, C, respectively. If cevian AD is drawn so that $BD = m$, $DC = n$ and $AD = d$, we have that $b^2m + c^2n = amn + d^2a$. (This is also often written $man + dad = bmb + cnc$, a form which invites mnemonic memorization, i.e. "A man and his dad put a bomb in the sink.")



- **Theorem** (can use if you know determinants) A triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) has an area equal to the absolute value of $\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$. For example the area of the triangle with vertices

$(3,5), (6,-3), (-2,-2)$ is $Area = \frac{1}{2}\begin{vmatrix} 3 & 6 & -2 \\ 5 & -3 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\left[1\begin{vmatrix} 6 & -2 \\ -3 & -2 \end{vmatrix} - 1\begin{vmatrix} 3 & -2 \\ 5 & -2 \end{vmatrix} + 1\begin{vmatrix} 3 & 6 \\ 5 & -3 \end{vmatrix}\right] =$
$\frac{1}{2}\{[6(-2)-(-2)(-3)] - [3(-2)-(-2)5] + [3(-3)-6\cdot 5]\} = \frac{1}{2}(-18-4-39) = 30.5$

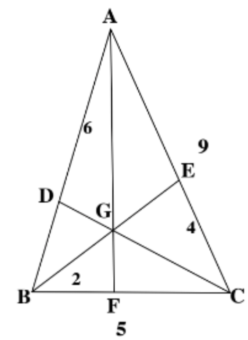
Problems

For the following problems, assume a calculator is not allowed unless stated.

Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. When the height of a triangle is quadrupled (made four times larger), its area increased by 2019. What is the area of the original triangle? [calculator allowed] (MSHSML 2019-20 2B #1)
2. In $\triangle ABC$ at the right, $AC = 9$ and $BC = 5$. Segments $\overline{BE}$, $\overline{CD}$, and $\overline{AF}$ are concurrent at G. If $BF = 2$, $CE = 4$, and $AD = 6$, determine exactly DB. [calculator allowed] (MSHSML 2018-19 2B #1)



Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. In $\triangle ABC$, $AB = 13$, $BC = 4$, and $CA = 15$. Cevian $\overline{AD}$ is drawn such that $CD = 1$. Determine exactly $[ADC]$.[1] [calculator allowed] (MSHSML 2019-20 2B #2)
2. What is the area of a triangle with side lengths 25, 25, and 48? [calculator allowed] (MSHSML 2018-19 2B #2)

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. In *Figure 3*, AB=5, AC=7, $BC = 4\sqrt{7}$, and $\overline{AM}$ is a median of $\triangle ABC$. Determine exactly AM. [calculator allowed] (MSHSML 2019-20 2B #3)

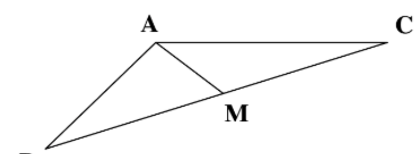


Figure 3

2. $\triangle ABC$ has an area of 16. If $AC = 8$ and median $\overline{BM}$ has a length of $\sqrt{17}$, determine exactly the perimeter of $\triangle ABC$. [calculator allowed] (MSHSML 2018-19 2B #3)

Problem #4 ("challenge"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

[1] Recall the notation $[ABC]$ indicates the area of the polygon (a triangle, in this example) ABC.

1. In *Figure 4*, $MN = 12$, $NP = 14$, $MP = 16$, $\overline{MR} \perp \overline{NP}$, and $\overline{MT}$ bisects $\angle NMP$. Determine exactly the area of $\triangle MRT$. [calculator allowed] (MSHSML 2019-20 2B #4)

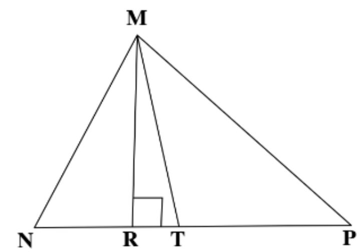


Figure 4

2. If two sides of a triangle have lengths 25 and 20 and the median to the third side has a length of 19.5, determine exactly the length of the third side. [calculator allowed] (MSHSML 2018-19 2B #4)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 (“challenge problem”; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge of other mathematical areas (algebra, for example). For more MSHSML Meet 2 Event B problems, see past exams, which date back to 1980-81.