Math Team Notes
Topic 2C: Trigonometry

Subtopics
Topic 2C, Trigonometry, includes the following subtopics.

2C	**Precalculus & Trigonometry: Trigonometry**
2C1	Functions of sums of angles and sums of functions of angles
2C2	Half-angle formulas and double-angle formulas
2C3	Power-reduction formulas
x	Not required: formulas for $\sin A + \sin B$, etc.

Notes

- **Pythagorean identities**
  These identities are derived from the right-triangle-based trigonometric function definitions (see Meet 1 Event C notes) and the Pythagorean Theorem. For example, for a right triangle with legs *opp* and *adj* and hypotenuse *hyp*, $opp^2 = adj^2 = hyp^2 \Rightarrow \frac{opp^2}{hyp^2} = \frac{adj^2}{hyp^2} = \frac{hyp^2}{hyp^2} \Rightarrow \sin^2\theta + \cos^2\theta = 1$.
  - $\sin^2\theta + \cos^2\theta = 1$
  - $\tan^2\theta + 1 = \sec^2\theta$
  - $\cot^2\theta + 1 = \csc^2\theta$
- **Angle-sum identities**
  - $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
  - $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
  - $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- **Double-angle identities**
  These are derived from the angle-sum identities. For example, $\sin(2\theta) = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$
  - $\sin(2\theta) = 2\sin\theta\cos\theta$
  - $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
  - $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$
- **Half-angle identities**
  - $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}}$
  - $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}}$
  - $\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$
- **Power-reduction identities**
  These are derived from the product identities. For example, $\sin^2\theta = \sin\theta\sin\theta = \frac{1}{2}[-\cos(\theta+\theta) + \cos(\theta-\theta)] = \frac{1}{2}(-\cos(2\theta) + 1) = \frac{1}{2}(1 - \cos(2\theta))$.
  - $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$
  - $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$
  - $\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$

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- **Even- and odd-function identities**
  These are best remembered by thinking how the function is graphed. The graph of an ***even function*** is symmetric with respect to the y-axis, and the graph of an ***odd function*** is symmetric with respect to the origin.
  - $\sin(-\theta) = -\sin\theta$ (odd)
  - $\cos(-\theta) = \cos\theta$ (even)
  - $\tan(-\theta) = -\tan\theta$ (odd)
- **Complementary-angle identities**
  - $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
  - $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
- **Product identities**
  These are derived from the angle-sum identities. For example, $\sin(A+B) + \sin(A-B) = (\sin A\cos B + \cos A\sin B) + (\sin A\cos B - \cos A\sin B) = 2\sin A\cos B$, so $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$.
  - $\sin A\sin B = \frac{1}{2}[-\cos(A+B) + \cos(A-B)]$
  - $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
  - $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
  - $\tan A\tan B = \frac{\cos(A-B) - \cos(A+B)}{\cos(A-B) + \cos(A+B)}$
- **Other multiples of $15°$ identities** (not required, though occasionally useful)
  These are derived from the angle-sum identities. For example, to find an expression for $\sin 15°$, note that $15° = 45° - 30°$ and evaluate $\sin 15° = \sin(45° - 30°) = \sin 45°\cos 30° + \cos 45°\sin 30° = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$.
  - $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4} = \cos 75°$
  - $\cos 15° = \frac{\sqrt{6}+\sqrt{2}}{4} = \sin 75°$
  - $\tan 15° = 2 - \sqrt{3}$ $\quad\quad$ $\tan 75° = 2 + \sqrt{3}$
- **Multiples of $18°$ identities** (not required, though occasionally useful)
  - $\sin 18° = \cos 72° = \frac{1-\sqrt{5}}{4}$
  - $\cos 36° = \sin 54° = \frac{1+\sqrt{5}}{4}$
- **Triple-angle identities** (not required, though occasionally useful)
  - $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
  - $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
  - $\tan 3\theta = \frac{\tan\theta(\tan^2\theta - 3)}{3\tan^2\theta - 1}$
- **Sum-to-product identities** (not required, though occasionally useful)
  - $\sin A \pm \sin B = 2\sin\frac{A\pm B}{2}\cos\frac{A\mp B}{2}$
  - $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$ $\quad\quad$ $\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
  - $\tan A \pm \tan B = \frac{\sin(A\pm B)}{\cos A\cos B}$

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Problems

For the following problems, assume a calculator is not allowed unless stated. Keep in mind that even when a calculator *is* allowed it may not be *necessary* or even *worthwhile* (because the brain is faster).

Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. If $\frac{\pi}{2} < B < \pi$ and $\sin B = \frac{5}{13}$, determine exactly $\sin(2B)$. (MSHSML 2019-20 2C #1)
2. Determine exactly the smallest possible positive degree measure for θ, given that $\tan 9\theta = 1$. (MSHSML 2018-19 2C #1)

Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. If $\pi < A < \frac{3\pi}{2}$ and $\sin A = -\frac{7}{25}$, determine exactly $\cos\frac{A}{2}$. (MSHSML 2019-20 2C #2)
2. Determine exactly the value of $\tan\frac{5\pi}{12}$. (MSHSML 2018-19 2C #2)

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. In $\triangle ABC$, $\cos(A + B) = \frac{2}{3}$ and $\sin(A + B) = \frac{3}{4}$. Determine exactly $\sin(2C)$. (MSHSML 2019-20 2C #3)
2. Determine exactly the smallest positive degree ind the values for x, given that $\tan x + \tan 47° = \sqrt{3} - \sqrt{3}\tan x \tan 47°$. (MSHSML 2018-19 2C #3)

Problem #4 ("challenge"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

1. Determine exactly the maximum value of the function $f(x) = \frac{2\sin^6 x \cdot \cos^2 x}{(\tan^2 x + 1)^2}$. (MSHSML 2019-20 2C #4)
2. Find the values of the constants A, B, and C that make the following equation an identity: $\sin x \sin\left(x + \frac{\pi}{3}\right) = A\sin 2x + B\cos 2x + C$. Express your answer as an ordered triple (A, B, C). (MSHSML 2018-19 2C #4)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge of other mathematical areas (algebra or geometry, for example). For more MSHSML Meet 2 Event C problems, see past exams, which date back to 1980-81.