Math Team Notes
Topic 2D: Analytic Geometry of Lines and Circles

Summary

The purpose of these notes is to support mathlete preparation for participation in Minnesota State High School Mathematics League Meet 2, Individual Event D: Algebra 2 & Analysis. The notes primarily address the newly introduced subtopics, and are therefore not comprehensive; mathletes are encouraged to review material beyond these notes (including notes for prior meets).

Subtopics

Topic 2D, Analytic geometry of lines and circles, includes the following subtopics.

2D	**Algebra 2 and Analysis: Analytic Geometry of Lines and Circles**
2D1	Slope, families of parallel, perpendicular, or coincident lines
2D2	Point-slope, slope-intercept, intercept, normal forms of a line
2D3	Intersections (as a solution of system of equations)

Notes

- **Definition** The ***slope*** of a line is the ratio of the vertical change (rise) to the horizontal change (run).
- **Definition** Two or more lines are ***parallel*** if they have the same slope.
- **Definition** Two lines are ***perpendicular*** if their slopes multiply to -1. Equivalently, two lines are perpendicular if their slopes are negative reciprocals of each other.
- **Definition** A ***family of lines*** is a set of lines that has something in common. There are three types:
  - A ***family of parallel lines*** is a set of lines that each have the same slope.
  - A ***family of coincident lines*** is a set of lines that each pass through a single point.
  - A ***family of perpendicular lines*** is a set of lines that are each perpendicular to a given line through a given point. In a plane, such a "family" consists of only one line. (However, the family becomes more interesting in dimensions greater than 2.)
- If (x_1, y_1) and (x_2, y_2) are the coordinates of any two distinct points on the same line, the slope m of that line may be calculated using the formula $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$.
- **Definition** A linear equation written in the form $y = mx + b$ is written in ***slope-intercept form***, where m is the slope of the line and b is the y-intercept. For example, in the equation $y = -\frac{2}{3}x + 4$, the slope is $-\frac{2}{3}$ and the y-intercept is 4.
- **Definition** A linear equation written in the form $ax + by = c$ is written in ***standard form***, where a, b, and c are real numbers and a and b are not both zero. Some textbooks define standard form slightly differently, as $ax + by + c = 0$. Sometimes standard form is termed ***general form***.
- To graph a linear equation written in standard form, rather than convert the equation to slope-intercept form it is often easier and quicker to graph using intercepts. For example, to graph the line $2x + 3y = 12$, find the x-intercept by setting $y = 0$ and solve for the x-intercept as $x = \frac{12}{2} = 6$, and find the y-intercept by setting $x = 0$ and solve for the y-intercept as $y = \frac{12}{3} = 4$. Then use these two points (the intercepts) to draw the line.
- **Definition** A linear equation written in the form $y - y_1 = m(x - x_1)$ is written in ***point-slope form***. Note the similarity with $m = \frac{y_2 - y_1}{x_2 - x_1}$.

- Given a slope and a point, one can use the point-slope form to write the slope-intercept form of the line. For example, a line with slope -2 that passes through the point $(10,12)$ may first be written as $y - 12 = -2(x - 10)$, and then rearranged to obtain $y = -2x + 32$.
- If you know the coordinates of two points on a line, you can use these to find the slope via $m = \frac{y_2 - y_1}{x_2 - x_1}$. Then you can substitute the value of m and one of the points in the point-slope form, then solve for y to write the equation in slope-intercept form.
- **Definition** A linear equation written in the form $\frac{x}{a} + \frac{y}{b} = 1$ is written in ***intercept form***, where $a, b \neq 0$. The x-intercept is a and the y-intercept is b; the line intercepts the x-axis at the point $(a, 0)$ and intercepts the y-intercept at the point $(0, b)$.
- **Definition** A linear equation written in the form $x \cos\theta + y \sin\theta = p$ is written in ***normal form***, where p is the (positive) distance of the normal (i.e., perpendicular) segment from the origin to the line, and θ is the oriented (signed) angle from the positive *x*-axis to the normal segment.
- To convert from standard form $ax + by = c$ to normal form, divide each coefficient by $\frac{c}{|c|}\sqrt{a^2 + b^2}$.[1]
- The equation of a circle with center (h, k) and radius r is $(x - h)^2 + (x - k)^2 = r^2$.
- An interesting analysis of circle-line intersections is given at http://mathworld.wolfram.com/Circle-LineIntersection.html

Problems

For the following problems, assume a calculator is not allowed unless stated. Keep in mind that even when a calculator *is* allowed it may not be *necessary* or even *worthwhile* (because the brain is faster).

Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. Determine exactly the point of intersection of the line defined by $f(x) = \frac{3x+2}{6}$ and the line defined by $f^{-1}(x)$. [calculator allowed] (MSHSML 2019-20 2D #1)
2. Calculate the slope of the line $8x + 11y - 13 = 0$. [calculator allowed] (MSHSML 2018-19 2D #1)

Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. Let l_1 be the line $5x - 4y = 9$ and l_2 be the line $10x - Ay = 2$, where A is a constant. There is one value for A such that $l_1 \parallel l_2$ and another value for A such that $l_1 \perp l_2$. Determine exactly the product of these two values of A. [calculator allowed] (MSHSML 2019-20 2D #2)
2. Determine exactly, in the form $Ax + By = C$, the equation of the line with a negative slope that contains the center and the y-intercept of the circle $(x - 4)^2 + (y - 5)^2 = 65$. [calculator allowed] (MSHSML 2018-19 2D #2)

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[1] Note this divisor can be written using the ***sign function*** as $\frac{c}{|c|}\sqrt{a^2 + b^2} = \operatorname{sgn}(c)\sqrt{a^2 + b^2}$, where

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0. \\ -1 & x < 0 \end{cases}$$

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. A circle defined by $x^2 + y^2 + 9x - 6y = C$ is tangent to the x-axis. Determine exactly the value of C. [calculator allowed] (MSHSML 2019-20 2D #3)
2. A regular octagon lies in the first quadrant with its center at (m, m), and one of its sides lies on the x-axis. Determine exactly the product of the slopes of all the diagonals with positive slopes. [calculator allowed] (MSHSML 2018-19 2D #3)

Problem #4 ("challenge"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

1. The four points $(a + 1, 1)$, $(2a + 1, 2)$, $(17, a)$, and $(10, b)$ are collinear and $a > 0$. Determine exactly the y-intercept of the line perpendicular to this through $(10, b)$. [calculator allowed] (MSHSML 2019-20 2D #4)
2. Two circles intersect at $(4,1)$ and $(7,2)$. One circle is centered on the x-axis while the other circle is centered on the y-axis. Determine exactly the area of the triangle formed by the line containing these two centers, the x-axis, and the y-axis. [calculator allowed] (MSHSML 2018-19 2D #4)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge of other mathematical areas (geometry, for example). For more MSHSML Meet 2 Event D problems, see past exams, which date back to 1980-81.