Math Team Notes
Topic 3A: Systems of Linear Equations in Two (or Occasionally in Three) Variables

Summary

The purpose of these notes is to support mathlete preparation for participation in Minnesota State High School Mathematics League Meet 3, Individual Event A: Algebra 1. The notes primarily address the following newly introduced subtopics, and are therefore not comprehensive; mathletes are encouraged to review material beyond these notes, such as notes for prior meets and various textbooks. (And problems. Do lots and lots of problems.)

Subtopics

Topic 3A, Linear equations in one unknown, includes the following subtopics.

3A Algebra 1: Systems of Linear Equations in Two (or on occasion three) Variables
3A1 Numeric and literal systems of equations
3A2 Relation to graphical procedures
3A3 Word problems leading to systems of equations
3A4 Systems of inequalities used to define a region in a plane
3A5 Determinants

Notes

- **Definition** a ***literal equation*** is an equation with two or more variables. For example, $2x + 3y = 6$ is a literal equation.
- **Definition** a ***system of equations*** is a set of two or more equations. Sometimes called a ***set of simultaneous equations***.
- **Definition** a ***literal system*** is a system of equations with two or more variables. Note that not every equation has to contain each variable. For example, the equations $x = 3$ and $x + y - 2 = 3$ form a literal system.
- Just as an inequality in a single variable may be used to describe an interval of values on a number line (for example, $x \geq 3$ is the region of the number line that includes and is to the right of 3), one or more inequalities in two variables may be used to describe a region of values on a coordinate plane.
- **Definition** A ***numeric equation*** is simply what we would call an equation, one in which the values of the solution are numbers. For example, $2x - 8 = 6$ is a (numeric) equation.
- **Definition** To ***solve*** a system of equation is to solve for each variable in the system. There are several methods for doing this, including solving by graphing (or the "*graphical method*"), the substitution method, the elimination method, and solving by determinants using Cramer's Rule.
- **Definition** The ***graphical method*** (or "*solving by graphing*")is a method for solving a system of equations (and/or inequalities) in which each equation (or inequality) is graphed in the appropriate "space" (typically a number line, a plane, or 3D space, as these are the easiest for humans to visualize). A calculator may be used to implement the graphical method, though the solution may only be approximate and not exact.
- **Definition** The ***substitution method*** is a method for solving a system of equations in which one equation is solved for one variable (often in terms of the other variable(s)), and the expression for that one variable is substituted into (each of) the other equation(s), which decreases by one the number of equations in the system. For example, to solve the system of equations $2x + 3y = 6$ and $x + y = 1$, one might first solve the second equation for x (obtaining $x = 1 - y$), substitute $1 - y$ for x in the first equation (obtaining $2(1 - y) + 3y = 6$), solve this equation for y (obtaining $y = 4$), substitute 4 for y in either of the original equations and then solve for x (obtaining $x = -3$). The solution is the ordered pair $(x, y) = (-3,4)$.
- **Tip** In using the substitution method, look for ways to avoid fractions. If a variable term has a coefficient of 1 or -1, solving that equation for that variable is quick and easy.

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- **Definition** The elimination method is a method for solving a system of equations in which one or more equations are multiplied by a factor (possibly a different factor for different equations) and then added together to eliminate a variable, which decreases by one the number of equations in the system. For example, to solve the system of equations $2x + 3y = 6$ and $x + y = 1$, one might first multiply the second equation by -2 (obtaining $-2x - 2y = -2$, add this equation to the first equation (obtaining $0x + y = 4$), solve this equation for y (obtaining $y = 4$), substitute 4 for y in either of the original equations and then solving for x (obtaining $x = -3$). The solution is the ordered pair $(x, y) = (-3,4)$.
- **Tip** In using the elimination method, look for ways to minimize the number of multiplications. If the coefficient of a variable in one equation is an integer multiple of the coefficient of the same variable in the other equation, then multiplying that second equation by the integer is quick and easy. Also, to minimize silly sign mistakes, I recommend avoiding subtracting equations; instead: multiply an equation by a *negative* number and *add* the equations.
- **Definition** A ***matrix*** is a rectangular array of numbers. The ***dimensions*** of a matrix are its number of rows and columns. A matrix with m rows and n columns is an $m \times n$ matrix (pronounced "m by n matrix"). Matrices (plural of matrix) are typically represented by capital letters. For example, the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ has $m = 2$ rows and $n = 3$ columns, and thus A is a 2×3 ("2 by 3") matrix.
- **Definition** A ***square matrix*** is a matrix that has the same number of rows and columns.
- **Definition** The ***determinant*** of a 2×2 square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to $ad - bc$. We write this as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. For example, $\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix} = (3)(-3) - (2)(4) = -9 - 8 = -17$. Note that the determinant is a real number, which may be positive, negative, or zero.
- **Definition** An ***element*** of a matrix is an entry in the matrix. For example, the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}$ has four elements: $a_{11} = 3, a_{12} = 2, a_{21} = 4$, and $a_{22} = -3$.
- **Theorem**: **Cramer's Rule** The solution of the linear system $ax + by = e$, $cx + dy = f$ is $x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, $y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$. For example, the solution of the linear system $3x + 2y = -1$, $4x - 3y = 10$ is $x = \frac{\begin{vmatrix} -1 & 2 \\ 10 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}} = \frac{(-1)(-3)-(2)(10)}{(3)(-3)-(2)(4)} = \frac{3-20}{-9-8} = \frac{-17}{-17} = 1$, $y = \frac{\begin{vmatrix} 3 & -1 \\ 4 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}} = \frac{(3)(10)-(-1)(4)}{-17} = \frac{30\mp4}{-17} = \frac{-34}{-17} = 2$. The solutions is the ordered pair $(x, y) = (1,2)$.
- **Tips** Learn Cramer's Rule! It is often the quickest and easiest way to solve a system of 2 linear equations. Note that the denominator matrix is the same for both x and y, so you need only calculate it once.
- **Definition** The ***minor*** of an element of a matrix is the matrix that results by removing the row and the column that contains the element. For example, the minor of element 1 in $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is $\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$. Note that the minor has one fewer row and one fewer column than the original matrix.
- **Theorem** The ***determinant*** of a 3×3 square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ can be determined via a process called ***expansion by minors***. To find the determinant via expansion by minors across the first row, multiply each element of the first row by its minor. Then subtract the middle product and add the final product. Thus $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$. For

example, the determinant of $\begin{vmatrix} 5 & 0 & 1 \\ -4 & 2 & -7 \\ 3 & -6 & 8 \end{vmatrix} = 5\begin{vmatrix} 2 & -7 \\ -6 & 8 \end{vmatrix} - 0\begin{vmatrix} -4 & -7 \\ 3 & 8 \end{vmatrix} + 1\begin{vmatrix} -4 & 2 \\ 3 & -6 \end{vmatrix} =$
$5(2 \cdot 8 - (-7)(-6)) - 0 + 1((-4)(-6) - 2 \cdot 3) = 5(-26) - 0 + (18) = -112.$

- **Tip** You may have wondered why the second term in the previous description and example is subtracted. Expansion by minors may be performed across any row or any column, but the sign of the term associated with a given minor is assigned according to this pattern: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$.
- **Tip** You *can* use Cramer's Rule to solve a system of 3 equations, but it takes some time.
- **Theorem** The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is the absolute value of $\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$.

Problems

For the following problems, assume a calculator is not allowed unless stated.

Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. Determine exactly the coordinates of the intersection of the lines $x + y = 15$ and $5x + 8y = 87$. (MSHSML 2019-20 3A #1)
2. Determine exactly the coordinates of the intersection of the lines $\frac{x}{5} + \frac{y}{2} = 1$ and $\frac{-3x}{4} + \frac{y}{2} = 1$. [calculator allowed] (MSHSML 2018-19 3A #1)
3. If $x + 2y = 9$ and $2x + y = 12$, what is the value of $x - y$? [calculator allowed] (MSHSML 2017-18 3A #1)
4. Determine exactly the area of the region in the first quadrant bounded by $\frac{x}{4} + \frac{y}{10} = 1$. [calculator allowed] (MSHSML 2016-17 3A #1)

Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. Four times Frank's favorite number is 25 more than Gabby's favorite number and four times Gabby's favorite number is 26 more than Frank's favorite number. Determine exactly the sum of Frank's and Gabby's favorite numbers. (MSHSML 2019-20 3A #2)
2. Determine exactly the ordered triple (x, y, z) that satisfies this system of equations:
   $7x + 2y - 4z = 19$
   $5x + 3y - 3z = 15$
   $5x - 3y + 3z = 15$
   [calculator allowed] (MSHSML 2018-19 3A #2)
3. If the following three lines intersect at a single point, what is the value of $b - a$? [calculator allowed] (MSHSML 2017-18 3A #2)
   $2x + y = 1$
   $3x - y = 4$
   $ax + by = 7$
4. Given $\begin{vmatrix} 2 & 9 \\ 3 & b \end{vmatrix} = 2$, determine exactly $\begin{vmatrix} 9 & 2 \\ b & 3 \end{vmatrix}$. [calculator allowed] (MSHSML 2016-17 3A #2)

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

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1. Layton has a rectangle. He triples the height and doubles the width and notes that the perimeter is now equal to 30. He does the same thing again to this new rectangle and notes that the perimeter is now 70. Determine exactly the perimeter of the original rectangle. (MSHSML 2019-20 3A #3)
2. Apples and melons are on sale at the local farmers' market. Elaine buys 10 apples and 5 melons, pays with $10.00 and receives change. Xi buys 5 apples and 10 melons, pays with $10.00 and also receives change. Elaine and Xi give me their change and I add 10 cents and buy 3 apples and 1 melon, receiving 2 cents in change. Jorge buys 20 apples and 20 melons with $25.00 and receives $1.00 in change. How much does each apple cost? [calculator allowed] (MSHSML 2018-19 3A #3)
3. Given that x, y, and z are positive integers, if (x_1, y_1, z_1) is a solution to the following system, determine the smallest possible value of $y_1 - x_1$. [calculator allowed] (MSHSML 2017-18 3A #3)
   $x + 2y + 3z = 54$
   $3x + 2y + z = 54$
4. Five years ago I was the age my brother is now. When I am fifty, my brother will be three less than twice the age he is now. How old am I? [calculator allowed] (MSHSML 2016-17 3A #3)

<u>Problem #4 ("challenge"; 2 points)</u>

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

1. If $w + x + y + 6z = 8.5$, and $w + x + 6y + z = -2.3$, and $w + 6x + y + z = -0.\overline{6}$, and $6w + x + y + z = 3.5$, determine exactly the value of $wxyz$. (MSHSML 2019-20 3A #4)
2. A pedestrian and a cyclist leave point A for point B simultaneously at 10:00. Reaching point B, the cyclist immediately turns around and, while returning to point A, passes the pedestrian at 10:20. When the cyclist reaches point A again, she immediately turns around and catches up to the pedestrian at 10:30. If the cyclist and the pedestrian each traveled at a uniform rate, what time will it be when the pedestrian finally reaches point B? [calculator allowed] (MSHSML 2018-19 3A #4)
3. How many distinct pairwise intersection points are there to the following set of equations? [calculator allowed] (MSHSML 2017-18 3A #3)
   $2x + 5y = 8$
   $3x + 6y = 9$
   $4x + 7y = 10$
   $5x + 8y = 11$
   $6x + 9y = 12$
   $7x + 10y = 13$
4. Find the <u>sum</u> of all positive integers a and b where $a > b$ and the determinant $\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & a \\ b & b & 1 \end{vmatrix} = -13$.
   [calculator allowed] (MSHSML 2016-17 3A #3)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge of other mathematical areas (algebra, for example). For more MSHSML Meet 3 Event A problems, see past exams, which date back to 1980-81.