Math Team Notes
Topic 3B: Polygonal Figures and Solids

Subtopics

Topic 3B, Polygonal Figures and Solids, includes the following subtopics.

3B Geometry: Polygonal Figures and Solids
3B1 Special quadrilaterals and regular polygons (including area formulas)
3B2 Intersecting diagonals
3B3 Circumscribed polygons and Ptolemy's Theorem
3B4 Polygonal prisms and pyramids (including surface area and volume)

Notes

- **Definition** A ***polygon*** is a figure that is bounded by line segments and lies in a plane; it is a two-dimensional figure.
- **Definitions** A ***triangle***, ***quadrilateral***, ***pentagon***, ***hexagon***, ***heptagon***, ***octagon***, ***nonagon***, and ***decagon*** have 3, 4, 5, 6, 7, 8, 9, and 10 sides, respectively. An $\boldsymbol{n}$**-gon** is a general term for a polygon with n sides which allows us to avoid creating (and remembering!) specific names for an infinite number of polygons. For example, a pentagon with 17 sides may be referred to as a 17-gon. It is acceptable to refer to a polygon with a name, even a well-known name, using n-gon terminology. For example, a triangle may be referred to as a 3-gon. Other names include ***dodecagon*** (12 sides) and ***icosagon*** (20 sides).
- **Definition** An ***altitude*** of any quadrilateral that has parallel sides is the distance between them. Also, ***altitude*** can refer to any perpendicular line segment that connects points on the lines of the parallel sides.
- **Definitions** A polygon is ***convex*** if, for each pair of points inside the polygon, the line segment connecting them lies entirely inside the polygon. Conversely, a polygon is ***concave*** if for a pair of points inside the polygon, the line segment connecting them does *not* lie entirely inside the polygon.
- **Definition** A ***diagonal*** of a polygon is a line segment that connects any two nonconsecutive vertices.
- **Theorem** The sum of the angles of a quadrilateral is $360°$.
- **Definition** A ***rectangle*** is a quadrilateral each of whose angles is a right angle.
- **Corollary** A quadrilateral is equiangular iff it is a rectangle.
- **Definition** A ***parallelogram*** is a quadrilateral whose opposite sides are parallel.
- **Theorem** The opposite sides and angles of a parallelogram are equal.
- **Theorem** The diagonals of a parallelogram bisect each other.
- **Theorem** A quadrilateral is a parallelogram if its opposite angles are equal.
- **Theorem** A quadrilateral is a parallelogram if two opposite sides are both parallel and equal.
- **Theorem** A quadrilateral is a parallelogram if its diagonals bisect each other.
- **Definition** A ***square*** is a quadrilateral all of whose sides and angles are equal.
- **Definition** A ***rhombus*** is a quadrilateral all of whose sides are equal.
- **Theorem** All rectangles are parallelograms.
- **Theorem** All rhombuses are parallelograms.
- **Theorem** The diagonals of a rectangle are equal.
- **Theorem** The diagonals of a rhombus are perpendicular.
- **Definition** A ***trapezoid*** is a quadrilateral that has exactly one pair of parallel sides. The parallel sides are called the ***bases*** of the trapezoid, and the non-parallel sides are called its ***legs***.
- **Definition** An ***isosceles trapezoid*** is a trapezoid whose legs are equal.
- **Theorem** The base angles of an isosceles trapezoid are equal.
- **Theorem** The diagonals of an isosceles trapezoid are equal.
- **Definition** A ***kite*** is a quadrilateral with two pairs of equal-length adjacent sides.
- **Theorem** A quadrilateral is a kite iff one diagonal is the perpendicular bisector of the other.
- **Theorem** A quadrilateral is a kite iff one diagonal is the angle bisector of the two angles it meets.

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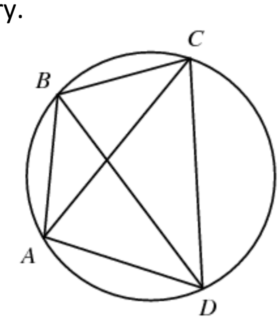
- **Theorem** A quadrilateral is a kite iff one diagonal is a line of symmetry, i.e., it divides the quadrilateral into two congruent triangles.
- **Theorem** The diagonals of a kite are perpendicular.
- **Definition** A ***polygonal region*** is the union of a polygon and its interior.
- **Definition** The ***area of a polygon*** is the measure of the region bounded by the polygon, i.e., its polygonal region.
- **Postulate** Every polygonal region has a positive number called its *area* such that (1) congruent triangles have equal areas, and (2) the area of a polygonal region is equal to the sum of the areas of its nonoverlapping parts.
- **Theorem** The area of a rectangle is the product of its base and altitude.
- **Corollary** The area of a square is the square of its side.
- [Triangle area theorems not included here – see Meet 2 Event B notes]
- **Theorem** The area of a parallelogram is the product of any base and corresponding altitude.
- **Theorem** The area of a trapezoid is half the product of its altitude and the sum of its bases.
- **Theorem** The area of a kite is $\frac{1}{2}pq$, where p and q are its diagonal lengths.
- **Theorem** The area of a kite is $ab\sin\theta$, where a and b are the lengths of two unequal sides and θ is the angle between unequal sides.
- **Definition** A polygon is ***cyclic*** iff there exists a circle that contains all of its vertices.
- **Definitions** A polygon is ***inscribed in a circle*** iff each vertex of the polygon lies on the circle. The circle is said to be ***circumscribed about the polygon***.
- **Theorem** Every triangle is cyclic.
- **Corollary** The perpendicular bisectors of the sides of a triangle are concurrent.
- **Theorem** A quadrilateral is cyclic iff a pair of its opposite angles are supplementary.
- **Ptolemy's Theorem** The sum of the products of the opposite side of a cyclic quadrilateral is equal to the product of its diagonals. Alternately: if ABCD is a cyclic quadrilateral, then $AB \times CD + AD \times BC = AC \times BD$ (see figure at right). Note that when ABCD is a rectangle, the result is the Pythagorean Theorem (which is helpful for remembering Ptolemy's Theorem).



- **Brahmagupta's Theorem** If a cyclic quadrilateral has perpendicular diagonals, then any line through their point of intersection that is perpendicular to a side of the quadrilateral will bisect the opposite side.
- **Brahmagupta's Area Theorem** The area of cyclic quadrilateral is $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where a, b, c, and d are the lengths of the sides of the quadrilateral and $s = \frac{a+b+c+d}{2}$ is its semiperimeter.
- **Definitions** A circle is **inscribed in a polygon** iff each side of the polygon is tangent to the circle. The polygon is said to be **circumscribed about the circle**. The circle is called the **incircle** of the polygon, and its center is called the **incenter** of the polygon.
- **Theorem** Every triangle has an incircle.
- **Corollary** The angle bisectors of a triangle are concurrent.
- **Definition** A ***regular polygon*** is a convex polygon that is both equilateral and equiangular.
- **Theorem** Every regular polygon is cyclic.
- **Definition** The ***center*** of a regular polygon is the center of its circumscribed circle.
- **Definition** A ***central angle*** of a regular polygon is an angle formed by radii drawn to two consecutive vertices.
- **Definition** An ***apothem*** (A-puh-them) of a regular polygon is a perpendicular line segment from its center to one of its sides.
- **Theorem** The perimeter p of a regular polygon having n sides of length s is $p = ns$.

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- **Theorem** The perimeter p of a regular polygon having n sides is $p = 2Nr$, where $N = n\sin\frac{180°}{n}$ and r is the radius. Note that as n gets larger, N approaches π (from below), as we would expect, since the circumference (perimeter) of a circle is $2\pi r$.
- **Theorem** The area A of a regular polygon with perimeter p and apothem a is $A = \frac{1}{2}pa$.
- **Theorem** The area A of a regular polygon having n sides is $A = Mr^2$, where $M = n\sin\frac{180°}{n}\cos\frac{180°}{n}$ and r is the radius.
- **Definition** The ***circumference*** of a circle is the limit of the perimeters of the inscribed regular polygons.
- **Definition** A ***polyhedron*** is a figure that is bounded by polygons and exists in space; it is a three-dimensional figure.
- **Definitions** A ***cube*** is a polyhedron with six square sides
- **Definitions** (formal) Suppose A and B are two parallel planes, R is a polygonal region in one plane, and l is a line that intersects both planes but not R. The solid made up of all segments parallel to line l that connect a point of region R to a point of the other plane is a **prism**. (informal) a solid "straight tube" with two ends that are congruent to each other and to any parallel cross-section.
- **Definitions** The two congruent faces of a prism are called its ***bases***. The rest of the faces of the prism are its ***lateral faces***. The edges in which the lateral faces intersect one another are its ***lateral edges***.
- **Definitions** If the lateral edges of a prism are perpendicular to the planes of its bases, the prism is a ***right prism*** and its lateral faces are rectangles. If the lateral edges of a prism are oblique (not perpendicular) to the planes of its bases, the prism is an ***oblique prism***.
- **Definitions** The ***lateral area*** of a prism is the sum of the areas of its lateral faces. The ***total area*** of a prism is the sum of its lateral area and the areas of its bases.
- **Definition** An ***altitude*** of a prism is a line segment that connects the planes of its bases and that is perpendicular to both of them.
- **Definition** A ***cross section*** of a geometric solid is the intersection of a plane and the solid.
- **Postulate** (***Cavalieri's Principle***) Consider two geometric solids and a plane. If every plane parallel to this plane that intersects one of the solids also intersects the other so that the resulting cross sections have equal areas, then the two solids have equal volumes. [insert diagram]
- **Postulate** The volume of any prism is the product of the area of its base and its altitude: $V = Bh$.
- **Corollary** The volume of a rectangular solid is the product of its length, width, and height: $V = lwh$.
- **Corollary** The volume of a cube is the cube of its edge: $V = e^3$.
- **Definitions** Suppose that A is a plane, R is a polygonal region in plane A, and P is a point not in plane A. The solid made up of all segments that connect P to a point of region R is a **pyramid**. The face of the pyramid that lies in this plane is its ***base***. The rest of its faces are the ***lateral faces*** and the edges in which they intersect each other are its ***lateral edges***. The lateral edges meet at the ***apex*** of the pyramid.
- **Definition** The ***altitude*** of a pyramid is the perpendicular line segment connecting the apex to the plane of its base (geometric object). It is also the length of this segment (numerical value).
- **Theorem** The volume of any pyramid is one-third of the product of the area of its base and its altitude: $V = \frac{1}{3}Bh$.

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Problems

For the following problems, assume a calculator is not allowed unless stated.

Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. A cube has side length of 3. A cylinder has a height of 3. They both have the same volume. Determine exactly the radius of the cylinder. [calculator allowed] (MSHSML 2019-20 3B #1)
2. If the areas of an equilateral triangle and a square are equal, determine exactly the ratio of the side of the square to the side of the triangle. Express your answer in the form $\left(\frac{a}{b}\right)^c$. [calculator allowed] (MSHSML 2018-19 3B #1)
3. The diagonals of a rhombus are 6 and 8. Calculate the area of the rhombus. [calculator allowed] (MSHSML 2017-18 3B #1)
4. Determine exactly the surface area of a sphere whose volume is 36π. [calculator allowed] (MSHSML 2016-17 3B #1)

Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. A trapezoid has an area of 36. One of the bases of the trapezoid has length 14 and the length of the other base is equal to the height of the trapezoid. Determine exactly the height of the trapezoid. [calculator allowed] (MSHSML 2019-20 3B #2)
2. Given a $3 \times 3 \times 3$ cube, a $1 \times 1 \times 1$ cube is cut out of the middle of each face. What is the surface area of the resulting solid? [calculator allowed] (MSHSML 2018-19 3B #2)
3. Determine exactly the area of an equilateral triangle if its circumscribed circle has a radius of 10. [calculator allowed] (MSHSML 2017-18 3B #2)
4. When the side lengths of a cube are all increased by 1, the surface area increases by 90. Calculate the volume of the original cube. [calculator allowed] (MSHSML 2016-17 3B #2)

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. Cyclic quadrilateral $ABCD$ has $AB = 12$, $BC = 8$, $CD = 5$, and $DA = 6$. If AC and BD are also integers, how long is AC? [calculator allowed] (MSHSML 2019-20 3B #3)
2. In *Figure 3*, $ABCD$ is a square whose side length is 32. $DMBN$ is a rhombus whose vertices lie on the diagonals of the square. If the area of the rhombus is 75% of the area of the square, determine exactly the length of $\overline{MN}$. [calculator allowed] (MSHSML 2018-19 3B #3)

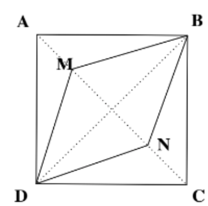

Figure 3

3. An isosceles trapezoid has the lengths of its congruent sides and smaller base equal to 1. Its diagonals and longer base have length x. Determine exactly the value of x. [calculator allowed] (MSHSML 2017-18 3B #3)

4. Points A, B, C, and D are located on circle O as shown in *Figure 3*. If chords $AB = BC = 3CD = 3AD$, determine exactly the ratio $AC:BD$. [calculator allowed] (MSHSML 2016-17 3B #3)

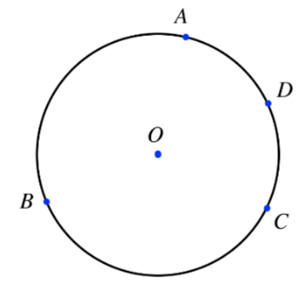

Figure 3

Problem #4 ("challenge"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

1. As shown in *Figure 4*, four congruent circles are inscribed in equilateral $\triangle ABC$ so that each circle has three points of tangency. The centers of these four circles are connected to form concave quadrilateral $MNOP$. If $[ABC] = 16\sqrt{3}$, what is $[MNOP]$? [calculator allowed] (MSHSML 2019-20 3B #3)

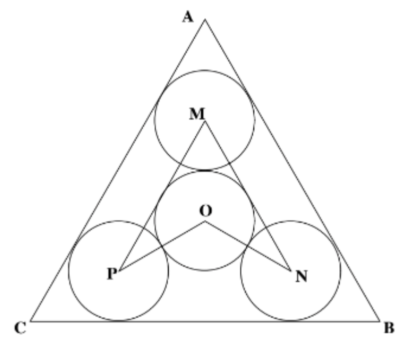

Figure 4

2. The solid shown in *Figure 4* has a rectangular base $BEFC$ with equilateral triangles ABC and DEF tilting toward each other. If M and N are the midpoints of $\overline{BC}$ and $\overline{EF}$, then $\measuredangle AMN = \measuredangle DNM = 60°$. If $BC = 43$ and $CF = 28$, determine exactly the volume of the solid. [calculator allowed] (MSHSML 2018-19 3B #4)

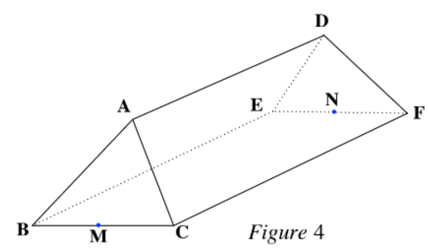

Figure 4

3. The area of quadrilateral $ABCD$ is 12 cm². On sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ are points F, K, M, and P, respectively, such that $AF:FB = 2:1$, $BK:KC = 1:3$, $CM:MD = 1:1$, and $DP:PA = 1:5$. Determine exactly the area of hexagon $AFKCMP$. [calculator allowed] (MSHSML 2017-18 3B #4)
4. Shown in *Figure 4* is a pyramid which has a square base with side lengths of 6, two sides of length 13, and two sides of length 11. Calculate the volume of this pyramid. [calculator allowed] (MSHSML 2016-17 3B #4)

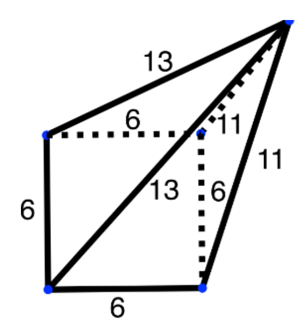

Figure 4

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge of other mathematical areas (algebra, for example). For more MSHSML Meet 3 Event B problems, see past exams, which date back to 1980-81.