**Math Team Notes**
**Topic 3C: Trigonometry**

**Summary**
The purpose of these notes is to support mathlete preparation for participation in Minnesota State High School Mathematics League Meet 3, Individual Event C: Trigonometry. The notes primarily address the following newly introduced subtopics, and are therefore not comprehensive; mathletes are encouraged to review material beyond these notes, such as notes for prior meets and various textbooks. (And problems. Do lots and lots of problems.)

**Subtopics**
Topic 3C, Trigonometry, includes the following subtopics.

**<u>3C Precalculus & Trigonometry: Trigonometry</u>**
**3C1** Law of Sines, Law of Cosines
**3C2** Inverse functions and their graphs
**3C3** Complex numbers in the complex plane, including both rectangular coordinates $(a, b) = a + bi$ and polar coordinates $(r, \theta) = r\cos\theta + i\sin\theta = r(\cos\theta + i\sin\theta) = \operatorname{cis}\theta$
**3C4** De Moivre's Theorem and the roots of unity

**Notes**
***<u>Law of Sines</u>***
The Law of Sines may be used to find an angle or side of *any* triangle (not just right triangles), provided other, sufficient side(s) or angle(s) are known.

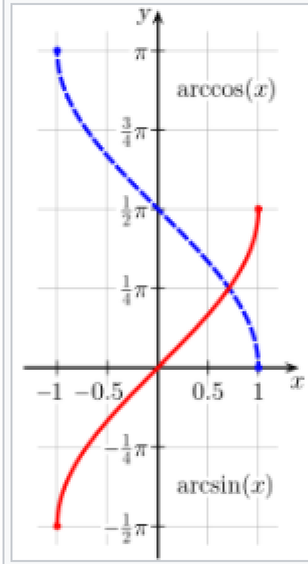
- Given $\Delta ABC$, with sides $a$, $b$, and $c$ across from angles $A$, $B$, and $C$, respectively, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. An equivalent form uses the reciprocals: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
- Use the Law of Sines when
  - the measures of two angles and any side of a triangle are known (AAS or ASA), or
  - the measures of two sides and an angle *opposite one of those sides* is known (SSA).
- **Tip** You might consider choosing the form for which the unknown is in the numerator. This saves time. In other words, use $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ when the unknown is an angle ($A$, $B$, or $C$), and use $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ when the unknown is a side ($a$, $b$, or $c$).

***<u>Law of Cosines</u>***
The Law of Cosines may be used to find an angle or side of *any* triangle (not just right triangles), provided other, sufficient side(s) or angle(s) are known.

- Given $\Delta ABC$, with sides $a$, $b$, and $c$ across from angles $A$, $B$, and $C$, respectively, $a^2 = b^2 + c^2 - 2bc\cos A$. Equivalent forms are $b^2 = a^2 + c^2 - 2ac\cos B$ and $c^2 = a^2 + b^2 - 2ab\cos C$. The equivalent form solved for the (cosine of the) angle is $\cos A = \frac{b^2+c^2-a^2}{2ab}$.
- Use the Law of Cosines when
  - The measures of two sides and *the included angle* are known (SAS), or
  - The measures of all three sides of a triangle are known (SSS)
- **Tip** You might consider choosing the form for which the unknown is by itself on one side of the equation. This saves time. In other words, use $a^2 = b^2 + c^2 - 2bc\cos A$ when the unknown is side $a$, and use $\cos A = \frac{b^2+c^2-a^2}{2ab}$ when the unknown is angle $A$.

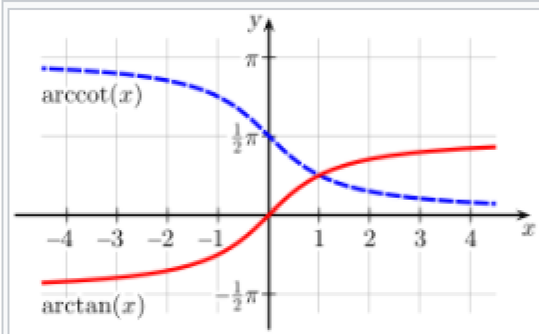
<ins>Inverse trigonometric functions and their graphs</ins>

- **Tip** Use the unit circle to evaluate trigonometric inverse values. For example, to describe all the values of $\sin^{-1}\frac{\sqrt{3}}{2}$, use the unit circle to see that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, then recall that adding multiples of $2\pi$ results in the same sine value, so that all the values of $\sin^{-1}\frac{\sqrt{3}}{2}$ are given by $\frac{\pi}{3} + 2\pi n$ and $\frac{2\pi}{3} + 2\pi n$, where $n$ is an integer.
- **Definition** The ***principal inverse sine function*** is $\mathrm{Sin}^{-1}\, a = \theta$, where $\mathrm{Sin}\,\theta = \mathrm{a}$. Its domain is $\{a| -1 \le a \le 1\}$ and its range is $\left\{\theta| -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$. See graph (Wikipedia, Inverse Trigonometric Functions).
- **Definition** The ***principal inverse cosine function*** is $\mathrm{Cos}^{-1}\, a = \theta$, where $\mathrm{Cos}\,\theta = \mathrm{a}$. Its domain is $\{a| -1 \le a \le 1\}$ and its range is $\{\theta|0 \le \theta \le \pi\}$. See graph (Wikipedia, Inverse Trigonometric Functions).



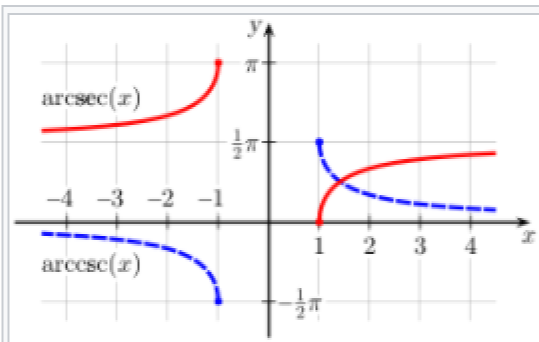
The usual principal values of the arcsin(x) (red) and arccos(x) (blue) functions graphed on the cartesian plane.

- **Definition** The ***principal inverse tangent function*** is $\mathrm{Tan}^{-1}\, a = \theta$, where $\mathrm{Tan}\,\theta = \mathrm{a}$. Its domain is $\{a| -\infty < a < \infty\}$ and its range is $\left\{\theta| -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}$. See graph (Wikipedia, Inverse Trigonometric Functions).
- **Definition** The ***principal inverse cotangent function*** **is** $\mathbf{Cot}^{-1}\, \boldsymbol{a} = \theta$, where $\mathrm{Cot}\,\theta = \mathrm{a}$. Its domain is $\{a| -\infty < a < \infty\}$ and its range is $\{\theta|0 < \theta < \pi\}$. See graph (Wikipedia, Inverse Trigonometric Functions)/



The usual principal values of the arctan(x) and arccot(x) functions graphed on the cartesian plane.

- **Definition** The ***principal inverse cosecant function*** is $\mathrm{Csc}^{-1}\, a = \theta$, where $\mathrm{Csc}\,\theta = \mathrm{a}$. Its domain is $\{a| -\infty < a \le 1\ \cup\ 1 \le a < \infty\}$ and its range is $\left\{\theta| -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \ne 0\right\}$. See graph (Wikipedia, Inverse Trigonometric Functions).
- **Definition** The ***principal inverse secant function*** is $\mathrm{Sec}^{-1}\, a = \theta$, where $\mathrm{Sec}\,\theta = \mathrm{a}$. Its domain is $\{a| -\infty < a \le 1\ \cup\ 1 \le a < \infty\}$ and its range is $\{\theta|0 \le \theta \le$



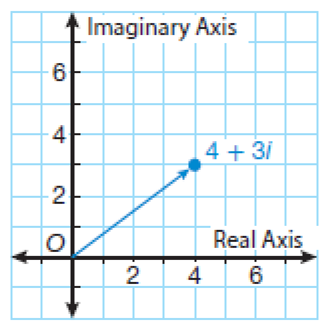
Principal values of the arcsec(x) and arccsc(x) functions graphed on the cartesian plane.

$\pi, \theta \neq \frac{\pi}{2}\}$. See graph (Wikipedia, Inverse Trigonometric Functions).

- **Note** The (non-principal) inverse trigonometric relations (which are *not* functions) have graphs that repeat above and below the principal inverse trigonometric function graphs. Because there is more than one $y$-value for each $x$-value, these are not functions.
- **Notation** In practice, the principal inverse trigonometric functions are often not capitalized, yet the limited range is understood. For example, $\mathrm{Sin}^{-1}\, a$ is often written as $\sin^{-1} a$, though it is understood that the restricted range is intended, allowing the relation to be a function.
- **Definition** The inverse trigonometric functions are also called the ***arcsine***, ***arccosine***, ***arctangent***, ***arccotangent***, ***arccosecant***, and ***arcsecant*** functions.

Complex Numbers in the Complex Plane

- **Definitions** An ***imaginary number*** is written in the form $bi$ where $b$ is a real number and $\boldsymbol{i}$ is the ***imaginary unit***. i is a solution of $x^2 = -1$. Because $i^2 = -1$, $i = \sqrt{-1}$. The square root of a negative number is an imaginary number. For example, $\sqrt{-16} = \sqrt{16 \cdot -1} = \sqrt{16}\sqrt{-1} = 4i$.
- **Definitions** A ***complex number*** is a number that can be written in the form $a + bi$, where $a$ and $b$ are real numbers. In a complex number, $a$ is called the ***real part*** and $bi$ is called the ***imaginary part***. If $b = 0$, then the imaginary part is 0 and the number is a real number. If $a = 0$ and $b \neq 0$, then the real part is 0 and the number is an imaginary number. Whereas an unknown real number is often given the name $x$, an unknown complex number is often given the name $z$.
- **Definition** The ***complex conjugate*** (or, when the context of complex numbers is understood, just ***conjugate***) of the complex number $z = a + bi$ is the complex number $\bar{z} = a - bi$. Note that $z\bar{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2 i^2 = a^2 - b^2(-1) = a^2 + b^2$.




- **Definitions** The ***complex plane*** consists of a plane demarcated by a pair of axes, the ***real axis*** (the horizontal, or $x$-axis) and the ***imaginary axis*** (the vertical, or $y$-axis).
- **Definition** The ***rectangular form*** of a complex number $z$ is $z = a + bi$, where $a$ and $b$ are real numbers and $i$ is the imaginary constant $i = \sqrt{-1}$.
- A complex number of the form $a + bi$ can be graphed with $a$ as the $x$-coordinate and $b$ as the $y$-coordinate. For example, the complex number $4 + 3i$ is graphed in the figure at right.
- **Definition** The ***magnitude*** $r$ of a complex number $z = a + bi$ is given by $r = |z| = |a + bi| = \sqrt{a^2 + b^2}$. Note $r \geq 0$. The magnitude corresponds to the length of the segment from the origin to the graph of the point on the complex plane. Note too that $r = |z| = \sqrt{z\bar{z}}$.
- **Definition** the ***angle*** $\theta$ of a complex number $z = a + bi$ is the counter-clockwise angle from the positive real axis and the ray from the origin to the corresponding point of the complex number in the complex plane. $\theta$ may be expressed in either degrees or radians.
- **Definition** The ***polar form*** of a complex number $z$ is $z = r\cos\theta + i(r\sin\theta)$, where $r$ is the magnitude of the complex number and $\theta$ is the angle of the complex number.
- **Notation** The notation $\operatorname{cis}\theta$ is shorthand for $\cos\theta + i\sin\theta$. Note the letters *c* (cosine), *i* (imaginary constant) and *s* (sine) in sequence. Thus the polar form of a complex number $z$ may be written compactly as $z = r\cos\theta + i(r\sin\theta) = r\operatorname{cis}\theta$.

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## Topic 3C: Trigonometry

**De Moivre's Theorem**

- **Theorem** Complex numbers of the form $(a + bi)^n$ for any real number $n$ (not just integers) are much easier to calculate in polar form. ***De Moivre's Theorem*** states

  $(r\cos\theta + ir\sin\theta)^n = r^n\cos n\theta + ir^n\sin n\theta$ or, more compactly: $(r\operatorname{cis}\theta)^n = r^n\operatorname{cis} n\theta$

  In words, the $n$th power of a complex number with magnitude $r$ and angle $\theta$ is a complex number with magnitude $r^n$ and angle $n\theta$.
- **Tip** De Moivre's Theorem may be used in reverse to find the roots of unity. (Unity here means the number 1.) You already know that the equation $x^2 = 1$ has two roots (because it is a polynomial equation of degree 2) and that the values of these roots are the solutions to $x = 1^{1/2} = \pm\sqrt{1} = \pm 1$. Another way to look at finding the roots of a complex number is to take the $n$th *principal* root (the real, positive root) of the magnitude and pair this with each of the $n$ angles of the form $\frac{m}{n}(360°)$, $m = 1,2,\dots,n-1$. (These may be thought of as the $n$ equispaced angles around the unit circle that always includes 0°.) So to find the two (square) roots of 1, the magnitude is the principal root $1^{1/2} = 1$ and the angles are $\frac{0}{2}(360°) = 0°$ and $\frac{1}{2}(360°) = 180°$, which yield the solutions $1\cos 0° + i1\sin 0° = 1 + 0i = 1$ and $1\cos 180° + i1\sin 180° = -1 + 0i = -1$.
- **Example** The former method looks unnecessarily difficult, and it is, for *square* roots of 1. The payoff is for roots other than square roots and for roots for real numbers other than unity. For example, to find the six 6th roots of 1, determine the magnitude as $1^{1/6} = 1$ and the angles as 0°, 60°,120°,180°,240°, and 300°, so that the roots are
  - $1^{1/6}\cos 0° + i1^{1/6}\sin 0° = 1(1) + 1(0)i = 1$
  - $1^{1/6}\cos 60° + i1^{1/6}\sin 60° = 1\left(\frac{1}{2}\right) + 1\left(\frac{\sqrt{3}}{2}\right)i = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
  - $1^{1/6}\cos 120° + i1^{1/6}\sin 120° = 1\left(-\frac{1}{2}\right) + 1\left(\frac{\sqrt{3}}{2}\right)i = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
  - $1^{1/6}\cos 180° + i1^{1/6}\sin 180° = 1(-1) + 1(0)i = -1$
  - $1^{1/6}\cos 240° + i1^{1/6}\sin 240° = 1\left(-\frac{1}{2}\right) + 1\left(-\frac{\sqrt{3}}{2}\right)i = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$
  - $1^{1/6}\cos 300° + i1^{1/6}\sin 300° = 1\left(\frac{1}{2}\right) + 1\left(-\frac{\sqrt{3}}{2}\right)i = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

  The six sixth roots of unity are thus $\pm 1$, $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, and $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Note that complex roots always come in complex-conjugate pairs, so it was actually unnecessary to actually *calculate* the final two roots above. Note too that it is helpful to draw the unit circle, mark the equispaced angles, and, when possible (as for sixth roots with an angle spacing of 60°, and generally for angles that are multiples of $30°$ or $45°$), calculate the cosines and sine values.
- **Example** To find the roots of a real number other than unity, multiply by the principal root of the real number. For example, to find the six sixth roots of 10, multiply the six sixth roots of unity determined previously by $10^{1/6}$ to obtain $\pm 10^{1/6}$, $10^{1/6}\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)$, and $10^{1/6}\left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)$. (The $10^{1/6}$ in latter two solution pairs may be distributed to obtain answers in the form $a + bi$.)

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## Problems

For the following problems, assume a calculator is not allowed unless stated.

Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. If $\sin^{-1}\left(\frac{1}{3}\right)=\theta$, determine exactly the value of $\cos(2\theta)$. (MSHSML 2019-20 3C #1)
2. $z=1+6i$. $w=z\cdot\bar{z}$, where $\bar{z}$ is the conjugate of $z$. Determine exactly the value of $w$. (MSHSML 2018-19 3C #1)
3. The exact solution to "Find all $x$, $0\le x<2\pi$, such that $\sin x=\frac{1}{3}$." is $x=\sin^{-1}\left(\frac{1}{3}\right)$ and $x=\pi-\sin^{-1}\left(\frac{1}{3}\right)$. Determine exactly the solution to "Find all $x$, $0\le x<2\pi$, such that $\sin x=-\frac{1}{3}$." (MSHSML 2017-18 3C #1)
4. Determine exactly $\cos^{-1}\left(\frac{-1}{2}\right)$. (MSHSML 2016-17 3C #1)

Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. In *Figure 2*, a hill rises at a constant angle of 23° from the horizontal. At the top of the hill stands a vertical flag pole that is 80 feet tall. A guy wire runs from the top of the flag pole to a point $P$ down the hill. At $P$ the guy wire makes a 30° angle with the hill. The length of the guy wire can ve written as $k\sin\theta$ for **acute** angle $\theta$. Determine exactly the ordered pair $(k,\theta)$. (MSHSML 2019-20 3C #2)

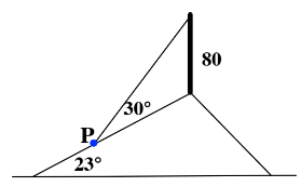

*Figure 2*

2. If $z=\text{cis}(30°)$, determine exactly the value of $z^3+\frac{1}{z^3}$. ($z=r\text{cos}(\theta)$ is shorthand notation for the complex number $r\cos\theta+r\sin\theta\, i$.) (MSHSML 2018-19 3C #2)
3. Determine exactly the value of $\cos\left(2\sin^{-1}\left(\frac{2}{3}\right)\right)$. (MSHSML 2017-18 3C #2)
4. Write as an ordered pair $(x,y)$ the point where $y\ =\ 3x-4$ intersects its inverse. (MSHSML 2016-17 3C #2)

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. In the complex plane, $A=\text{cis}(220°)$ and $B=\text{cis}(40°)$. If $A^k$ lies in the first quadrant and $A^k$, the origin, and $B$ are all collinear, what is the least positive integer value of $k$? (MSHSML 2019-20 3C #3)
2. If $\cos(\arctan x)=x$, then $x^2$ can be expressed exactly in the form $\frac{a+\sqrt{b}}{2}$. Calculate $a+b$. (MSHSML 2018-19 3C #3)
3. $N=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ and $M=\frac{-3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}i$. What imaginary number, written in the form $0+bi$, is on $\overline{MN}$ in the complex plane? (MSHSML 2017-18 3C #3)
4. In $\triangle ABC$, $AB=20\sqrt{3}$, $m\angle CAB=45°$, $m\angle ACB=60°$. Determine exactly $AC$. (MSHSML 2016-17 3C #3)

Problem #4 ("challenge"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

1. In $\triangle ABC$, $D$ lies on $\overline{AB}$ such that $AD=2$ and $DB=4$. $E$ lies on $\overline{AC}$ such that $AE=3$ and $EC=1$. If $DE=4$, determine exactly $CB$. (MSHSML 2019-20 3C #4)
2. Determine exactly, in terms of radians, the value of $\arctan(2-\sqrt{3})+\arctan(2+\sqrt{3})+\arctan(\sqrt{3})$. (MSHSML 2018-19 3C #4)

3. ($z = r\cos(\theta)$ is shorthand notation for the complex number $r\cos\theta + r\sin\theta\, i$.) $A = 0\,\text{cis}(0)$, $B = 1\,\text{cis}\left(-\frac{\pi}{12}\right)$, $D = 1\,\text{cis}\left(\frac{\pi}{4}\right)$, and $C = r\,\text{cis}(\theta)$. Determine exactly $(r, \theta)$ such that $ABCD$ forms a rhombus in the complex plane. (MSHSML 2017-18 3C #4)
4. In isosceles $\triangle MNP$, $MN = MP$, $NP = 1$, and $\cos M + \cos N + \cos P = 1.18$. If all sides have integer lengths, determine exactly the perimeter of $\triangle MNP$. (MSHSML 2016-17 3C #4)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge of other mathematical areas (algebra, for example). For more MSHSML Meet 3 Event C problems, see past exams, which date back to 1980-81.