Math Team Notes
Topic 3D: Exponents and Logarithms

Summary

The purpose of these notes is to support mathlete preparation for participation in Minnesota State High School Mathematics League Meet 3, Individual Event D: Algebra 2 & Analysis. The notes primarily address the following newly introduced subtopics, and are therefore not comprehensive; mathletes are encouraged to review material beyond these notes, such as notes for prior meets and various textbooks. (And problems. Do lots and lots of problems.)

Subtopics

Topic 3D, Exponents and logarithms, includes the following subtopics.

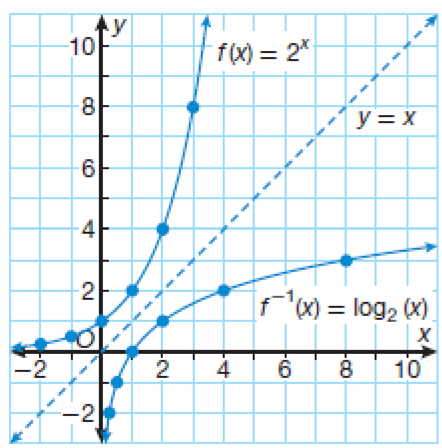
3D	**Algebra 2 & Analysis: Exponents and Logarithms**
3D1	Use of rational and negative exponents
3D2	Simplifying expressions involving radicals
3D3	Solving equations involving radicals
3D4	Use of logarithms; identities involving logarithms
3D5	Solving logarithmic equations
3D6	Relationships between logarithms of different bases

Notes

- **Definition** Exponents of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$, are called ***rational exponents***. They are sometimes called ***fractional exponents***.
- **Definition** Radical expressions of the form $\sqrt[n]{a}$ are called ***nth roots***. In this form n is the ***index*** and a is the ***radicand***. Similar to square roots, $\sqrt[n]{a} = b$ if $b^n = a$. The nth root of a real number a may be written in terms of rational exponents as $a^{\frac{1}{n}} = a^{1/n} = \sqrt[n]{a}$.
- **Theorem** (***Rational Exponent Property***) A real number a raised to a rational exponent may be written as $a^{\frac{m}{n}} = a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$. When evaluating an expression with a numerical exponent, you should generally take the (denominator) root first, and then raise the result to the power of the numerator. For example, $64^{2/3} = \left(64^{1/3}\right)^2 = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16$. It may be helpful to remember that in Minnesota, the legal driving age is $64^{2/3}$.
- **Theorem** (***Product Property of nth Roots***) For $a > 0$ and $b > 0$, $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.
- **Theorem** (***Quotient Property of nth Roots***) For $a > 0$ and $b > 0$, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.
- **Theorem** (***Rational exponent properties***) The properties of integer exponents hold true for rational exponents. For all nonzero real numbers a and b and integers m and n,
  - **Negative Exponent Property:** $a^{-n} = \frac{1}{a^n}$ (Note also: $\frac{1}{a^{-n}} = a^n$)
  - **Zero Exponent Property:** $a^0 = 1$
  - **Product of Powers Property:** $a^m \cdot a^n = a^{m+n}$
  - **Quotient of Powers Property:** $\frac{a^m}{a^n} = a^{m-n}$
  - **Power of a Power Property:** $(a^m)^n = a^{m \cdot n}$
  - **Power of a Product Property:** $(ab)^m = a^m \cdot b^m$
  - **Power of a Quotient Property:** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
- **Tip** When entering a rational exponent on a calculator, place the rational exponent within parentheses so that the entire rational expression is treated as an exponent. For example,

entering `9^1/2` will return the (probably unintended) value 4.5 (because, following the order of operations, $9^1 = 9$, and $\frac{9}{2} = 4.5$), while entering `9^(1/2)` will return the (probably intended) value 3 (because $9^{1/2} = 3$).

- **Definition** A ***logarithm*** is the exponent that is applied to a specified base to obtain a given value. Every exponential equation has a logarithmic form and vice versa. The exponential equation $b^x = a$ has the corresponding logarithmic equation $\log_b a = x$. A helpful pair to remember (and one that may be easily checked on a scientific calculator) is $10^2 = 100$ and $\log_{10} 100 = 2$. Again, these equations are equivalent. One may write $10^2 = 10 \Leftrightarrow \log_{10} 100 = 2$, where the symbol "⇔" may be read as "if and only if" or "is equivalent to."
- **Definition** If the base is not stated, it is assumed to be 10, i.e., $\mathbf{log}\,\boldsymbol{x} = \log_{10} x$.
- **Definition** The ***natural logarithm*** is the logarithm with base e, where ***e*** is the ***Euler constant***, $e \approx 2.71828$…. It is typically written as $\mathbf{ln}\,\boldsymbol{x}$, so that $\ln x = \log_e x$.
- **Theorem** $\log_b 1 = 0$ for any base $b \neq 0$. This is because $b^0 = 1$ for any $b \neq 0$.
- **Theorem** (***Inverse Property of Logarithms***) $b^{\log_b x} = x$ and $\log_b b^x = x$.
- **Theorem** (***Change of Base Property***) For $a > 0$ and $a \neq 1$ and any base b such that $b > 0$ and $b \neq 1$, $\log_b x = \frac{\log_a x}{\log_a b}$. The change of base property is often used to simplify expressions as well as determine a numerical value of a logarithm for a base that is not explicitly supported by a calculator. Calculators typically support only the base 10 logarithm and the base e logarithm (i.e., natural logarithm). For example, to find the exact value of $\log_4 8$, use the change of base property as $\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2}$. As another example, to find an approximate value of $\log_3 10$, use the change of base property as $\log_3 10 = \frac{\log_{10} 10}{\log_{10} 3} \approx \frac{1}{0.477121} \approx 2.096$. As a check, we can be satisfied that the result is between 2 ($= \log_3 2^3 = \log_3 9$) and 3 ($= \log_3 3^3 = \log_3 27$), as we would expect, since $2 = \log_3 9 < \log_3 10 < \log_3 27 = 3$.
- **Theorem** (***Product Property of Logarithms***) For any positive numbers m, n, and b ($b \neq 1$), $\log_b mn = \log_b m + \log_b n$.
- **Theorem** (***Quotient Property of Logarithms***) For any positive numbers m, n, and b ($b \neq 1$), $\log_b \frac{m}{n} = \log_b m - \log_b n$.
- **Theorem** (***Power Property of Logarithms***) For any real number p and positive numbers a and b ($b \neq 1$), $\log_b a^p = p \log_b a$.
- **Graphing** The logarithmic function is the inverse of the exponential function. Thus the graphs of the corresponding functions are reflected across the line $y = x$. For example, if $f(x) = 2^x$, then its inverse is $f^{-1}(x) = \log_2 x$. Recall that when a graph is reflected across the line $y = x$, the domain and range of each function are switched. So if the point $(0,1)$ is on the graph of $f(x)$, then the point $(1,0)$ is on the graph of $f^{-1}(x)$.



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Topic 3D: Exponents and Logarithms

Problems

For the following problems, assume a calculator is not allowed unless stated.

Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. Determine exactly the value of $\log_3 15 + \log_3 81 - \log_3 5$. (MSHSML 2019-20 3D #1)
2. Determine exactly the value of $\left(\frac{1}{64}\right)^{-\frac{1}{1}} + \left(\frac{1}{64}\right)^{-\frac{1}{2}} + \left(\frac{1}{64}\right)^{-\frac{1}{3}} + \left(\frac{1}{64}\right)^{-\frac{1}{6}}$. (MSHSML 2018-19 3D #1)
3. Determine exactly the value of x: $3\log_x 16 = 4$. (MSHSML 2017-18 3D #1)
4. For what x value will $4\log_3 x = 4$? (MSHSML 2016-17 3D #1)

Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. The solutions to the equation $2(\log x)^2$ are 10^m and 10^n. Determine exactly the product mn. (MSHSML 2019-20 3D #2)
2. Determine exactly the value of $\log_{12} 24 + \log_{12} 72$. (MSHSML 2018-19 3D #2)
3. For $\frac{2^{9x}4^{x-1}}{16^{x^2}} = \frac{1}{32}$, x has two solutions, a and b. Determine $a + b$. (MSHSML 2017-18 3D #2)
4. The expression $\frac{3(3^{2n-1})+9^{n-1}}{27^{\frac{2n}{3}-1}}$ can be written in the form $3^a + 3^b$. Determine exactly the sum $a + b$. (MSHSML 2017-18 3D #2)

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. Find the sum of all the solutions to the equation $(x^2 + 5x + 5)^{x^2-10x+21} = 1$. (MSHSML 2019-20 3D #3)
2. Let $Q = \log_3 15$. If the number $\log_3 375$ can be determined exactly in the form $a \cdot Q + b$, for some integers a and b, determine a and b. (MSHSML 2018-19 3D #3)
3. Determine exactly the value of x: $x\log_3 x = 18$. (MSHSML 2017-18 3D #3)
4. If $x > 2y > 0$ and $2\log(x - 2y) = \log x + \log y$, determine $\frac{x}{y}$ exactly. (MSHSML 2016-17 3D #3)

Problem #4 ("challenge"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

1. Two positive numbers x and y satisfy the equation $2 + \log_2 x = 5 + \log_5 y = \log(x + y)$. Determine exactly the value of $\frac{1}{x} + \frac{1}{y}$. (MSHSML 2019-20 3D #4)
2. Let $b = 3^{25}$. Determine exactly the value of x, $x \neq 1$, given that $\sqrt{\log_b x} = \log_{\sqrt{b}} x + \log_b \sqrt{x}$. (MSHSML 2018-19 3D #4)
3. Determine exactly all the values of x for which $\log_x\left(\log_{3x}\left(\log_{6x}(2^{12}3^{12}4^6 8^4 9^6 27^4)\right)\right) = 0$. (MSHSML 2017-18 3D #4)
4. The solution set of all x values for which $\log_4 x - \log_4 2 + \log_4(x - 4) \leq 2$ can be written as $a < x \leq b$. Determine exactly the values of a and b. (MSHSML 2017-18 3D #4)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge of other mathematical areas (algebra, for example). For more MSHSML Meet 3 Event D problems, see past exams, which date back to 1980-81.