Math Team Notes
Topic 4A: Algebraic Manipulation

Summary
The purpose of these notes is to support mathlete preparation for participation in Minnesota State High School Mathematics League Meet 3, Individual Event A: Algebra 1. The notes primarily address the following newly introduced subtopics, and are therefore not comprehensive; mathletes are encouraged to review material beyond these notes, such as notes for prior meets and various textbooks. (And problems. Do lots and lots of problems.)

Subtopics
Topic 4A, Algebraic manipulation, includes the following subtopics.

4A Algebra 1: Algebraic Manipulation
4A1 Factoring (including x^3+y^3 and x^3-y^3
4A2 Sums, products, quotients of rational expressions
4A3 Solving equations (including radical equations) involving these skills, but ultimately solvable by factoring or the quadratic formula (but no complex roots)
4A4 Rational exponents
4A5 Functional notation and variational dependencies (direct and inverse variation)

Notes
Factoring
- **Definition** to ***factor*** an algebraic expression is to replace it with an equivalent product of two or more simpler expressions. The ability to factor a polynomial expression of degree n into two or more products of polynomials each with degree less than n is useful in solving polynomial equations and graphing polynomial functions.
- **Theorem** Common polynomial factorizations include:
  1. $a^2-b^2=(a+b)(a-b)$ Note: a^2+b^2 cannot be factored
  2. $a^2+2ab+b^2=(a+b)^2$
  3. $a^2-2ab+b^2=(a-b)^2$
  4. $a^3+b^3=(a+b)(a^2-ab+b^2)$
  5. $a^3-b^3=(a-b)(a^2+ab+b^2)$
- **Definitions** Polynomials are named by their degree. Here are some common names:
  - A polynomial of degree 0 is a **constant**
  - A polynomial of degree 1 is a **linear** polynomial
  - A polynomial of degree 2 is a **quadratic** polynomial
  - A polynomial of degree 3 is a **cubic** polynomial
  - A polynomial of degree 4 is a **quartic** polynomial
  - A polynomial of degree 5 is a **quintic** polynomial
  - In general, a polynomial of degree n is an ***nth-degree polynomial***

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- **Definition** When an integer n is a factor of another integer N, we say ***n* divides *N*** and write $\boldsymbol{n|N}$. When a number n is not a factor of an integer N, we say ***n* does not divide *N*** and write $\boldsymbol{n \nmid N}$.
- **Tip** To factor quickly and accurately, it is helpful to know how to determine whether or not an integer n is a factor of a larger integer N quickly and accurately. It helps to be familiar with a set of what are called "divisibility" rules. You undoubtedly already know some of them. Here is sample list.

n	Rule for determining if n is a factor of N (i.e., n divides N, or $n|N$)
2	$2|N$ if and only if (iff) N is even (N ends in 0, 2, 4, 6, or 8) • Examples: (a) 2|216 because 6 is even; (b) $2 \nmid 97$ because 7 is not even.
3	$3|N$ iff 3 divides the sum of the digits of N • Examples: (a) 3|687 because 6+8+7 = 21, and 3|21; (b) $3 \nmid 701$ because 7+0+1=8 and $3 \nmid 8$. • Note: $3|N$ when N is comprised of 3 consecutive digits, because $n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)$. (It works if the 3 digits are spaced by 2 or 3 also, like 953 or 147.)
4	$4|N$ iff 4 divides the number formed by the last 2 digits of N above a multiple of 100. • Examples: (a) 4|908 because 908-900=8 and 4|8; (b) $4 \nmid 554$ because 554-500=54 and $4 \nmid 54$. • Note: This works because 4|100.
5	$5|N$ iff 5 divides the last (ones) digit of N, i.e., the ones digit of N is 0 or 5 • Examples: (a) 5|885 because 885 ends in 5; (b) $5 \nmid 329$ because $5 \nmid 9$.
6	$6|N$ iff N is divisible by 2 and 3 • Examples: (a) 6|972 because 972 is even, and 9+7+2=18 and 2|18; (b) $6 \nmid 329$ because $3 \nmid 329$.
7	Determining divisibility by 7 is a little more difficult than any of the other number less than 10. Here are two rules. Choose the one you prefer. 1st rule: $7|N$ iff the following procedure ends in a 0 or 7: (1) remove the ones digit and double it (2) subtract it from the remaining number, (3) repeat until the number is one digit. • Examples: (a) $7|12264$ because $1226 - 8 = 1218$, $121 - 16 = 105$, and $10 - 10 = 0$, (b) $7 \nmid 94$ because $9 - 8 = 1$ and $1 \neq 0$ or 7. 2nd rule: Beginning with the ones digit (on the right), multiply each digit in turn by 1, 3, 2, 6, 4, 5, (memorize this) and then back to 1, 3, 2, …., until all digits have been multiplied. Sum the products. $7|N$ iff 7 divides the sum of the products. • Examples: (a) 7|12264 because $4(1) + 6(3) + 2(2) + 2(6) + 1(4) = 4 + 18 + 4 + 12 + 4 = 42$ and $7|42$, (b) $7 \nmid 94$ because $4(1) + 9(3) = 4 + 27 = 31$ and $7 \nmid 31$. • Note: How to remember 132645? Maybe 13 (half the number of letters in alphabet), 26 (the number of letters in alphabet), 45 (number of minutes in a half in professional soccer)?
8	$8|N$ iff 8 divides the number formed by the last 3 digits of N above a multiple of 200. • Examples: (a) $8|12944$ because $12944 - 12800 = 144$ and $8|144$; (b) $8 \nmid 554$ because $554 - 400 = 154$ and $8 \nmid 154$. • Note: This works because 8|200 (and note also $8 \nmid 100$).
9	$9|N$ iff 9 divides the sum of the digits of N • Examples: (a) 3|657 because 6+5+7 = 18, and 9|18; (b) $9 \nmid 701$ because 7+0+1=8 and $9 \nmid 8$. • Note: $9|N$ when N is a two-digit number and the two digits add to 9, like 18, 27, 36, ….
10	$10|N$ iff the ones digit of N is 0 • Examples: (a) 10|123,450 because 123,450 ends in 0; (b) $10 \nmid 10101$ because 10101 ends in 1.
11	$11|N$ iff 11 divides the alternating sum of the digits of N • Examples: (a) $11|2728$ because $2 - 7 + 2 - 8 = -11$, and $11| - 11$; (b) $11 \nmid 701$ because $7 - 0 + 1 = 8$ and $11 \nmid 8$. • Note: $11|N$ when N is has an even number of digits that come in pairs, like 99 or 447733, because the subtraction within each pair gives 0, several zeros sum to zero, and $11|0$.

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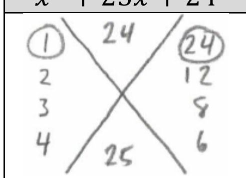
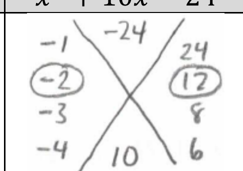
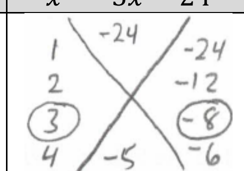
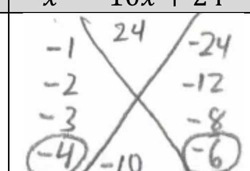
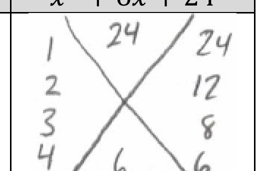
- **Method** To factor a quadratic polynomial of the form $x^2 + bx + c$ (note that the x^2 coefficient is 1), one can use an organizational method I call the ***Big X***. The time to explain the algorithm is much greater than the time to perform the algorithm. Once you get the hang of it, I think you'll like it.
    1. Write a (big) X on your paper. This divides your workspace into four regions. In the top region, write the constant coefficient c. In the bottom region, write the x coefficient b.
    2. In the left region at the top, write the number 1, and in the right region at the top, write the number $|c|$. (For the moment, we ignore the sign of c.) In the left region, identify the next factor of $|c|$ that is greater than 1 and write it beneath the 1, and in the right region, write its corresponding factor below the $|c|$. Continue in this way until you have written all factor pairs of $|c|$. In the left region, from top to bottom, you will have an increasing sequence of positive numbers beginning with 1, and in the right region, from top to bottom, you will have a decreasing sequence of positive numbers beginning with $|c|$, and each corresponding pair will multiply to $|c|$.
    3. If the value of c is negative, then exactly one of its two factors must be negative. Check the sign of b. If b is positive (see Example 2, below), then its smaller (in absolute value) factor is negative. Because of the way we organized the factors, all the smaller (in absolute value) factors are in the left region, so place a negative sign in front of each factor in the left region. On the other hand, if b is negative (see Example 3, below), then its larger (in absolute value) factor is negative. Because of the way we organized the factors, all the larger (in absolute value) factors are in the right region, so place a negative sign in front of each factor in the right region.
    4. If the value of c is positive, then its two factors are both positive, or its two factors are both negative. Check the sign of b. If b is positive (see Example 1, below), then the two (same-sign) factors add to a positive number, so they must both be positive, and you do not have to change any signs. On the other hand, if b is negative (see Example 4, below), then the two (same-sign) factors add to a negative number, so they must both be negative. Place a negative sign in front of each factor in both the left and right regions.
    5. At this point, all the factor pairs multiply to c (not necessarily $|c|$). Identify and circle the pair that adds to b. Let's say the pair of numbers are r and s. Then the original polynomial can be factored as $x^2 + bx + c = (x + r)(x + s)$. (Remember r and/or s may be negative.)
    6. If there is no factor pair that adds to b (see Example 5, below), then due to the completeness of the Big X algorithm you have demonstrated that the original polynomial $x^2 + bx + c$ *cannot be factored*. If you are attempting to solve a quadratic equation such as $x^2 + bx + c = 0$, you may complete the square or use the quadratic formula.
- **Example** Five examples of factoring a quadratic polynomial of the form $x^2 + bx + c$, along with the completed Big X and the factored form, are shown below:

1. Factor $x^2 + 25x + 24$	2. Factor $x^2 + 10x - 24$	3. Factor $x^2 - 5x - 24$	4. Factor $x^2 - 10x + 24$	5. Factor $x^2 + 6x + 24$
				
$(x + 1)(x + 24)$	$(x - 2)(x + 12)$	$(x + 3)(x - 8)$	$(x - 4)(x - 6)$	Not factorable

- **Method** the ***Box Method and the Big X*** is a method of factoring a quadratic polynomial of the form $ax^2 + bx + c$ in which $a \neq 1$. A video example is available here. Note: if $a = -1$, factor out a -1 first and then apply it at the end, obtaining $ax^2 + bx + c = -(px + r)(qx + s)$, where $p, q > 0$.

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Rational Expressions

- **Definition** A ***rational expression*** $\frac{f(x)}{g(x)}$ is the quotient of two polynomials $f(x)$ and $g(x)$, where $g(x) \neq 0$ and $g(x)$ is of degree ≥ 1. (If $g(x)$ is of degree 0, then the expression reduces to a single polynomial.) A rational expression is undefined for values of the variable that cause the denominator to equal zero.
- **Definition** A value for which the denominator of a rational function is undefined is called an ***excluded value***. Excluded values are not in the domain of the rational function.
- **Definition** A rational expression is ***simplified*** when there are no common factors, other than 1, in the numerator and denominator. Identify any excluded values prior to simplifying the expression.
- **Method** To multiply rational expressions, multiply the numerators and multiply the denominators.
- **Method** To simplify a rational expression, factor and then divide out all common factors.
- **Method** To evaluate an algebraic expression for given values of the variables, substitute the given values, and then simplify the numerical expressions.
- **Method** To divide by a rational expression, multiply by its reciprocal, just like you divide by a fraction: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, where b, c, and d are all nonzero.
- **Method** To add or subtract rational expressions,
  1. Find the least common denominator (LCD) of all the expressions.
  2. Modify each expression by multiplying both numerator and denominator by a factor that results in the denominator becoming the LCD.
  3. Add the numerators to form a new numerator in standard form and place over the LCD as denominator.
  4. Decide whether you wish to express the numerator and denominator both in standard form or express them both in factored form and convert as necessary.

Rational Exponents

- **Definition** Exponents of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$, are called ***rational exponents***. They are sometimes called ***fractional exponents***.
- **Definition** Radical expressions of the form $\sqrt[n]{a}$ are called ***nth roots***. In this form n is the ***index*** and a is the ***radicand***. Similar to square roots, $\sqrt[n]{a} = b$ if $b^n = a$. The nth root of a real number a may be written in terms of rational exponents as $a^{\frac{1}{n}} = a^{1/n} = \sqrt[n]{a}$.
- **Theorem** (***Rational Exponent Property***) A real number a raised to a rational exponent may be written as $a^{\frac{m}{n}} = a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$. When evaluating an expression with a numerical exponent, you should generally take the (denominator) root first, and then raise the result to the power of the numerator. For example, $64^{2/3} = \left(64^{1/3}\right)^2 = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16$. It may be helpful to remember that in Minnesota, the legal driving age is $64^{2/3}$.
- **Theorem** (***Product Property of nth Roots***) For $a > 0$ and $b > 0$, $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.
- **Theorem** (***Quotient Property of nth Roots***) For $a > 0$ and $b > 0$, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.
- **Theorem** (***Rational exponent properties***) The properties of integer exponents hold true for rational exponents. For all nonzero real numbers a and b and integers m and n,
  - **Negative Exponent Property:** $a^{-n} = \frac{1}{a^n}$ (Note also: $\frac{1}{a^{-n}} = a^n$)
  - **Zero Exponent Property:** $a^0 = 1$
  - **Product of Powers Property:** $a^m \cdot a^n = a^{m+n}$

- **Quotient of Powers Property:** $\frac{a^m}{a^n} = a^{m-n}$
- **Power of a Power Property:** $(a^m)^n = a^{m \cdot n}$
- **Power of a Product Property:** $(ab)^m = a^m \cdot b^m$
- **Power of a Quotient Property:** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

- **Tip** When entering a rational exponent on a calculator, place the rational exponent within parentheses so that the entire rational expression is treated as an exponent. For example, entering `9^1/2` will return the (probably unintended) value 4.5 (because, following the order of operations, $9^1 = 9$, and $\frac{9}{2} = 4.5$), while entering `9^(1/2)` will return the (probably intended) value 3 (because $9^{1/2} = 3$).

Variational Dependencies (Direct and Inverse Variation)

- **Definitions** When the statement of a problem says that ***A varies directly as B*** or that ***A is directly proportional to B***, the equation $A = kB$ is implied, where k is a constant. The relationship is called di***rect variation***. The constant k in the equation is called the ***constant of variation***. For example, circumference varies directly as radius: $C = 2\pi r$, where the constant of variation is 2π.
- **Definitions** If the product of two variables is a constant, then the equation is an ***inverse variation***. The equation may be expressed as $xy = k$ or $y = \frac{k}{x}$. Both x and y are variables and k is a nonzero constant. The term k is also referred to as the ***constant of variation***.
- **Note** Here are some differences between direct and inverse variation.

Direct Variation	Inverse Variation
As x increases, y increases.	As x increases, y decreases.
As x decreases, y decreases.	As x decreases, y increases.
The ratio $\frac{y}{x}$ is a constant.	The product yx is a constant.

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Problems

For the following problems, assume a calculator is not allowed unless stated.

Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. Determine exactly the value of $\frac{9^3}{3^9}$. (MSHSML 2019-20 4A #1)
2. If $p^2 = 2020 + q^2$ and $p = 10 + q$, compute $p + q$. (MSHSML 2018-19 4A #1)
3. Determine exactly the value of $\frac{1}{\sqrt{8}}$. (MSHSML 2017-18 4A #1)
4. Simplify the expression $\left(\sqrt{6} + \sqrt{24}\right)^2$. (MSHSML 2016-17 4A #1)

Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. Determine exactly the solution to $\frac{3x+2}{x+5} - \frac{8x+6}{3x+15} = 1$. (MSHSML 2019-20 4A #2)
2. If a and b are positive real numbers and $\frac{a^2+b^2}{\frac{1}{a^2}+\frac{1}{b^2}} = 10$, determine exactly the value of $\frac{a^3+b^3}{\frac{1}{a^3}+\frac{1}{b^3}}$. (MSHSML 2018-19 4A #2)
3. The equation $\frac{\sqrt{x^2-7}}{2} = \sqrt{\frac{x+4}{2}}$ has two solutions, a and b. What is the value of $a + b$? (MSHSML 2017-18 4A #2)
4. The solution set of $\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2}} < \sqrt{8}$ can be written in the form $x > a$. Determine a exactly. (MSHSML 2016-17 4A #2)

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. Determine exactly the solutions to $x^2 - \sqrt{x^2 - 9} = 21$. (MSHSML 2019-20 4A #3)
2. Determine exactly the value for x, given that $x + 2\sqrt{x} = 2$. (MSHSML 2018-19 4A #3)
3. $\sqrt{6 - 2\sqrt{5}}$ can be written in the form $a + b\sqrt{c}$, where a, b, and c are integers and c has no square factors. Determine the ordered triple (a, b, c). (MSHSML 2017-18 4A #3)
4. Given that $\frac{6}{x^2-1} + \frac{b}{x-1} + \frac{c}{x+1} = \frac{4}{x+1}$, determine exactly the values of b and c. (MSHSML 2016-17 4A #3)

Problem #4 ("challenge"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

1. If $4^x + 6^x = 9^x$, determine exactly the value of $\left(\frac{3}{2}\right)^x$. (MSHSML 2019-20 4A #4)
2. Let $P = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{100}{101}$ and $Q = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{101}{100}$. Determine exactly the value of $P + Q$. (MSHSML 2018-19 4A #4)
3. Determine exactly the value of $\frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}$. (MSHSML 2017-18 4A #4)
4. For $0 < a < b$, if $a^3 = ka - 841$ and $b^3 = kb - 841$, determine exactly the value of $(ab)(a + b)$. (MSHSML 2016-17 4A #4)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge of other mathematical areas (algebra, for example). For more MSHSML Meet 4 Event A problems, see past exams, which date back to 1980-81.