**Math Team Notes**
**Topic 4B: Circular Figures and Solids**

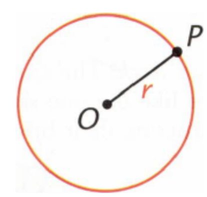

**Summary**
The purpose of these notes is to support mathlete preparation for participation in Minnesota State High School Mathematics League Meet 4, Individual Event B: Geometry. The notes primarily address the following newly introduced subtopics, and are therefore not comprehensive; mathletes are encouraged to review material beyond these notes, such as notes for prior meets and various textbooks. (And problems. Do lots and lots of problems.)


**Subtopics**
Topic 4B, Circular Figures and Solids, includes the following subtopics.

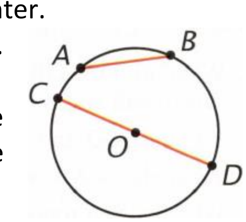
<u>**4B Geometry: Circular Figures and Solids**</u>
**4B1** Central, inscribed, tangential, and exterior angles
**4B2** Power of a point (chords, secants, tangents)
**4B3** Interior and exterior tangents of two circles
**4B4** Intercepted arcs
**4B5** Areas of circles, sectors, circular segments
**4B6** Cylinders, cones, and spheres (including volume and surface area)

**Notes**
All figures in these notes are from *Geometry*, 3rd edition, by Harold R. Jacobs (2003), with the exception of those in the <u>Power of a Point</u> section, which are from *The Art of Problem Solving Volume I: The Basics*, 7th edition, by Sandor Lehoczky and Richard Rusczyk (2015).

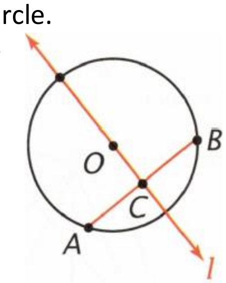
<u>Circles, Radii, and Chords</u>




- **Definition** A ***circle*** is the set of all points in a plane that are at a given distance from a given point in the plane. A circle does not contain its center, and it does not include its interior. The center is often designated by the point $O$. See figure.
- **Definition** A ***radius*** (geometrical object) of a circle is a line segment that connects the center of the circle to any point on it. *The* ***radius*** (number) of a circle is the length of one of these line segments. See figure.
- **Corollary** All radii of a circle are equal.
- **Definition** Circles are ***concentric*** iff they lie in the same plane and have the same center.
- **Definition** A ***chord*** of a circle is a line segment that connects two points of the circle. See figure.
- **Definition** A ***diameter*** (geometrical object) of a circle is a chord that contains the center. *The* ***diameter*** (number) of a circle is the length of one of these chords. See figure.




- **Theorem** If a line through the center of a circle is perpendicular to a chord, it also bisects the chord.
- **Theorem** If a line through the center of a circle bisects a chord that is not a diameter, it is also perpendicular to the chord.
- **Theorem** The perpendicular bisector of a chord of a circle contains the center of the circle.
- **Tip** The preceding three theorems are related and can be remembered as follows. See figure. Given that any two of the following statements are true, then all three statements are true:
    - The line $l$ contains the center of the circle.
    - The line $l$ Is perpendicular to the chord $\overline{AB}$.
    - The line $l$ bisects chord $\overline{AB}$.



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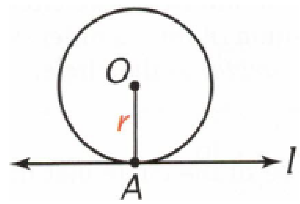
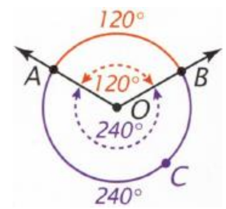
Tangents

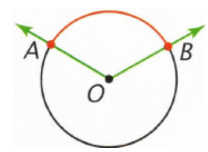



- **Definition** A ***tangent*** to a circle is a line in the plane of the circle that intersects the circle in exactly one point. For example, in the figure, line $l$ is a tangent line.
- **Theorem** If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.
- **Theorem** If a line is perpendicular to a radius at its outer endpoint, it is tangent to the circle.

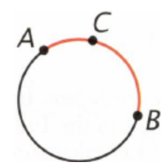
Central Angles and Arcs




- **Definition** A ***semicircle*** is half a circle.
- **Definition** A ***minor arc*** is an arc whose length is less than that of the semicircle of which it is part.[1] A minor arc is usually named by just two points, called its ***endpoints***. For example, in the figure $\widehat{AB}$ describes a minor arc.
- **Definition** A ***major arc*** is an arc whose length is greater than that of the semicircle it contains. A major arc (and a semicircle) is named with three letters, the middle letter naming a third point on the arc. For example, in the figure, $\widehat{ACB}$ describes a major arc.

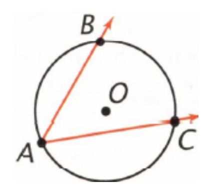



- **Definition** A ***central angle*** of a circle is an angle whose vertex is the center of a circle. For example, in the figure, $\angle AOB$ is a central angle.
- **Definition** A ***reflex angle*** is an angle whose measure is more than $180°$.
- **Definition** The ***degree measure*** of an arc is the measure of its central angle. The degree measure of $\widehat{AB}$ is designated $m\,\widehat{AB}$.

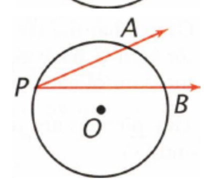



- **Arc Addition Postulate** If $C$ is on $\widehat{AB}$, then $m\,\widehat{AC} + m\,\widehat{CB} = m\,\widehat{ACB}$. See figure.
- **Theorem** In a circle, arcs with equal degree measure have equal-length chords. In other words that are more catchy but less precise: In a circle, equal arcs have equal chords.

Inscribed Angles




- **Definition** An ***inscribed angle*** of a circle is an angle whose vertex is on a circle and whose sides each intersect the circle in another point. For example, in the figure, $\angle BAC$ is an inscribed angle.



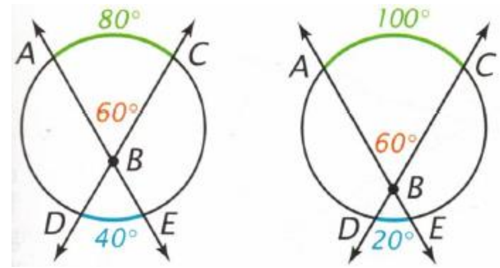

- **Theorem** An inscribed angle is equal in measure to half its intercepted arc. For example, in the figure, $m\angle APB = m\,\widehat{AB}$.
- **Corollary** Inscribed angles that intercept the same arc are equal.
- **Corollary** An angle inscribed in a semicircle is a right angle.

[1] A minor arc is also called a ***baby arc*** (doo-doo-doo-doo-doo-doo!). No, not really.

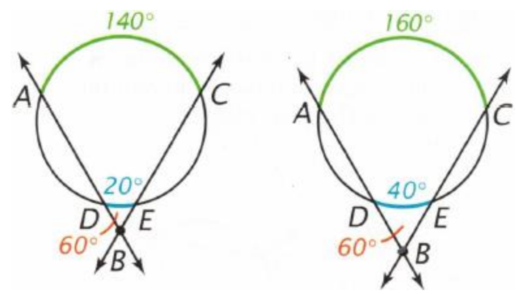
# Topic 4B: Circular Figures and Solids

## Secant Angles

- **Definition** A ***secant*** is a line that intersects a circle in two points.
- **Definition** A ***secant angle*** is an angle whose sides are contained in two secants of a circle so that each side intersects the circle in at least one point other than the angle's vertex. When the intersection is outside the circle, it is also called an ***exterior angle*** of a circle.
- **Theorem** A secant angle whose vertex is inside a circle is equal in measure to half the sum of the arcs intercepted by it and its vertical angle. For example, in the left figure, $m\angle ABC = \frac{1}{2}\left(m\,\widehat{AC} + m\,\widehat{DE}\right) = \frac{1}{2}(80 + 40) = 60°$, and in the right figure, $m\angle ABC = \frac{1}{2}\left(m\,\widehat{AC} + m\,\widehat{DE}\right) = \frac{1}{2}(100 + 20) = 60°$.

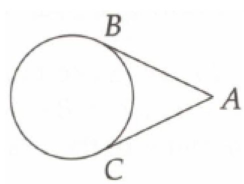


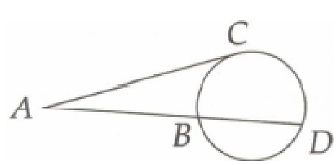

- **Theorem** A secant angle whose vertex is outside a circle is equal in measure to half the difference of its larger and smaller intercepted arcs. For example, in the left figure, $m\angle ABC = \frac{1}{2}\left(m\,\widehat{AC} - m\,\widehat{DE}\right) = \frac{1}{2}(140 - 20) = 60°$, and in the right figure, $m\angle ABC = \frac{1}{2}\left(m\,\widehat{AC} - m\,\widehat{DE}\right) = \frac{1}{2}(160 - 40) = 60°$

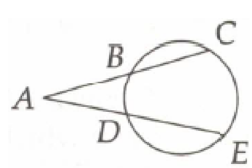


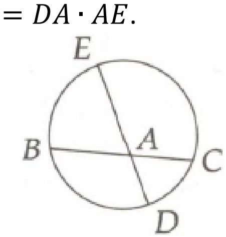

## Power of a Point

- **Power of a Point Theorem** Given a point $P$ and a line through $P$ which intersects a circle in two points $A$ and $B$, the product $PA \cdot PB$ is the same for any choice of line.
- **Definition** If a line is tangent to a circle, then any segment of the line having the point of tangency as one of its endpoints is a ***tangent segment*** to the circle. For brevity, it is sometimes called a ***tangent***.
- **Definition** If a line is a secant line of a circle, then any segment of the line with one endpoint exterior to the circle and one endpoint on the circle, with the second intersecting point in between, is a ***secant segment***. For brevity, it is sometimes called a ***secant***.
- **Note** The Power of a Point Theorem has four distinct cases. These may be stated with corollaries.
- **Corollary** The tangent segments to a circle from an external point are equal. For example, in the first figure below, $AB = AC$.
- **Corollary** If a tangent segment and a secant segment intersect outside a circle at point $P$, the tangent segment intersecting the circle at point $A$ and the secant segment intersecting the circle at points $B$ and $C$, then $PA^2 = PB \cdot PC$. For example, in the second figure below, $AC^2 = AB \cdot BD$.
- **Corollary** If two secant lines intersect outside a circle at point $P$, one secant line intersecting the circle at points $A$ and $B$ and the other secant line intersecting the secant line at points $C$ and $D$, then $PA \cdot PB = PC \cdot PD$. For example, in the third figure below, $AB \cdot AC = AD \cdot AE$.
- **Corollary** If two chords $AB$ and $CD$ intersect in a circle at point $P$, then $PA \cdot PB = PC \cdot PD$. In other words, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord. For example, in the fourth figure below, $BA \cdot BC = DA \cdot AE$.









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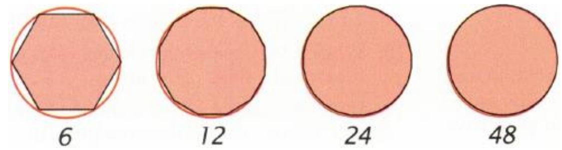
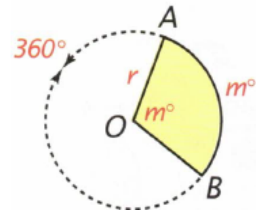
The Area of a Circle

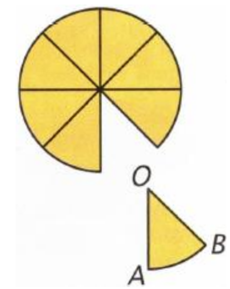



- **Definition** The ***area*** of a circle is the limit of the areas of the inscribed regular polygons. See figure.
- **Theorem** If the radius of a circle is $r$, its area is $\pi r^2$.

Sectors and Arcs

- **Definition** A ***sector*** of a circle is a region bounded by an arc of the circle and the two radii to the endpoints of the arc. (Like a slice of pizza.) See figure at far right.
- **Theorem** If $r$ is the radius of a circle, then the area $A$ of a sector of the circle with central angle (or intercepted arc) $\theta$ in degrees is given by $A = (\text{fraction of circle}) \cdot (\text{area of circle}) = \left(\frac{\theta}{360}\right)(\pi r^2)$. See figure at near right. If the central angle (or intercepted arc) $\theta$ is in radians, then $A = (\text{fraction of circle}) \cdot (\text{area of circle}) = \left(\frac{\theta}{2\pi}\right)(\pi r^2)$.

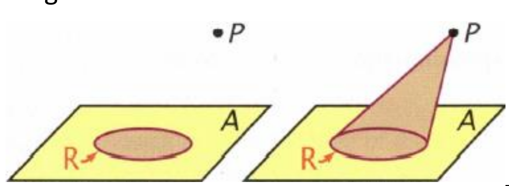


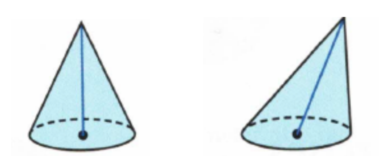




- **Theorem** If $r$ is the radius of a circle, then the length $l$ of a sector of the circle with central angle (or intercepted arc) $\theta$ in radians is given by $l = (\text{portion of circle})(\text{circumference of circle}) = \left(\frac{\theta}{2\pi}\right)(2\pi r)$. If the central angle (or intercepted arc) $\theta$ is in degrees, then $l = (\text{portion of circle}) \cdot (\text{circumference of circle}) = \left(\frac{\theta}{360}\right)(2\pi r) = \left(\frac{\pi}{180}\right) r\theta$. If the central angle (or intercepted arc) $\theta$ is in radians, then $l = (\text{fraction of circle}) \cdot (\text{circumference of circle}) = \left(\frac{\theta}{2\pi}\right)(2\pi r) = r\theta$.

Cones

- **Definition** a ***circular region*** is the union of a circle and its interior. In other words, a circular region is a “filled-in” circle. A circular region is sometimes called a ***disk***.
- **Definitions** Suppose that $A$ is a plane, $R$ is a circular region in plane $A$, and $P$ is a point not in plane $A$. The solid made up of all segments that connect $P$ to a point of a region $R$ is a ***cone***. The circular region is called the ***base*** of the cone. The point $P$ is called the ***apex*** of the cone. The curved surface is called the ***lateral surface*** of the cone. The line segment connecting the apex of the cone to the center of its base is called its ***axis***. A cone is either ***right*** or ***oblique***, depending on whether its axis is perpendicular or oblique to its base. See figures.



A right cone An oblique cone

- **Tip** A cone is a solid, and it contains the circular region $R$, the lateral surface, and all the points “inside.” If you were eating an ice cream cone, and you licked off the top ice cream so that the ice cream surface were flat across the top, then the rim of the cone would be the *circle*, the rim with the visible surface of the ice cream would be the *circular region*, the part you hold would be the *lateral surface*, and everything together (including the nonvisible ice cream inside) would be the *cone*.
- **Theorem** The volume of a cone is one-third the product of the area of its base and its altitude: $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$. This applies to both right and oblique cones.

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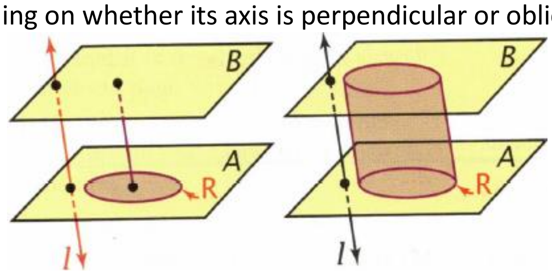
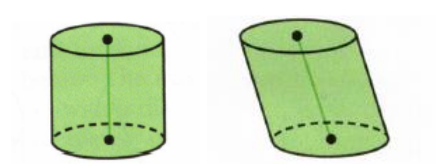
- **Tip** A cone is the circular counterpart to a pyramid. Use this to remember the volumes of each. The volume of a pyramid (with a polygon base) is $V = \frac{1}{3}Bh$, and the volume of a cone is $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$.
- **Definition** The ***height*** of a cone is the (perpendicular-to-the-plane) distance from the apex to the plane containing the base. Note that in a right cone, the height is the length of the axis.
- **Definition** The ***slant height*** of a right cone is the length of a segment from the apex to a point on the edge of the circular region that makes up the base. Note that slant height is not defined for oblique cones.
- **Theorem** The slant height of a right cone is the hypotenuse of the right triangle with legs radius of the base and height of the pyramid: $l = \sqrt{r^2 + h^2}$
- **Theorem** The lateral surface area of a right cone is one-half the product of the perimeter of the base and the slant height: $LSA = \frac{1}{2}pl = \frac{1}{2}(2\pi r)l = \pi rl = \pi r\sqrt{r^2 + h^2}$.
- **Theorem** The surface area of a right cone is the sum of the area of its base and the area of its lateral surface: $SA = B + LSA = B + \frac{1}{2}pl = \pi r^2 + \pi rl = \pi r^2 + \pi r\sqrt{r^2 + h^2}$. The "short" formula is $SA = \pi r^2 + \pi rl$.
- **Tip** A right cone is the circular counterpart to a pyramid. Use this to remember the surface area of each. The surface area of a pyramid (with a polygon base) is $SA = B + LSA = B + \frac{1}{2}pl$, and the surface area of a cone (with circular base) is $SA = B + LSA = B + \frac{1}{2}pl = \pi r^2 + \pi rl = \pi r^2 + \pi r\sqrt{r^2 + h^2}$. The "short" formula is $SA = \pi r^2 + \pi rl$, but it is helpful to understand where it comes from and how to determine $l$ from $r$ and $h$ so that you may determine the surface area from the given information.

Cylinders

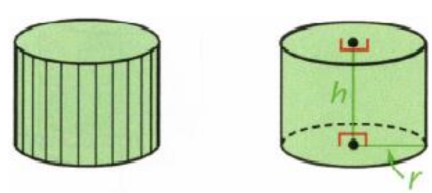
- **Definition** Suppose that $A$ and $B$ are two parallel planes, $R$ is a circular region in one plane, and $l$ is a line that intersects both planes but not $R$. The solid made up of all segments parallel to line $l$ that connect a point of region $R$ to a point of the other plane is a ***cylinder***. Each of the two flat surfaces (each is a circular region) is a ***base*** of the cylinder. The curved surface is the ***lateral surface*** of the cylinder. The line segment connecting the centers of its basis is the ***axis*** of the cylinder. A cylinder is either ***right*** or ***oblique***, depending on whether its axis is perpendicular or oblique to its base. See figures.



A right cylinder An oblique cylinder

- **Theorem** The volume of a cylinder is the product of the area of its base and its altitude: $V = Bh = \pi r^2 h$. See figure. Note this applies to both right and oblique cylinders.
- **Tip** A cylinder is the circular counterpart to a prism. Use this to remember the volumes of each. The volume of a prism (with polygon bases) is $V = Bh$, and the volume of a cylinder is $V = Bh = \pi r^2 h$.




- **Theorem** The surface area of a cylinder is the sum of the areas of the bases and the lateral surface area: $SA = 2B + LSA = 2\pi r^2 + ph = 2\pi r^2 + 2\pi rh$. Note this applies to both right and oblique cylinders.
- **Tip** A cylinder is the circular counterpart to a prism. Use this to remember the surface areas of each. The volume of a prism (with polygon bases) is $V = Bh$, and the volume of a cylinder is $V = Bh = \pi r^2 h$.

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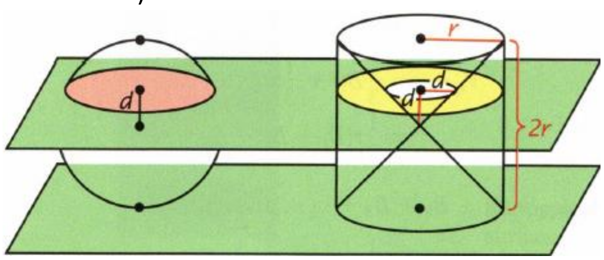
Spheres

- **Definitions** A ***sphere*** is the set of all points in space that are at a given distance from a given point. Note that this definition is like that of a circle except that the words "in space" replace "in a plane." As a result, words like ***center***, ***radius***, and ***diameter*** have the same meanings for spheres as they do for circles.
- **Theorem** The volume of a sphere is $\frac{4}{3}\pi$ times the cube of its radius: $V = \frac{4}{3}\pi r^3$.
- **Theorem** The surface area of a sphere is $4\pi$ times the square of its radius: $SA = 4\pi r^2$.
- **Tip** The expression for the volume of a sphere seems puzzling. Why $\frac{4}{3}$? The following explanation may help. The volume of a sphere of radius $r$ is equal to the volume between a right circular cylinder with a base of radius $r$ and a height $2r$ minus two right cones with a base of radius $r$ and a height of $r$: $V_{cylinder} - V_{two\ cones} = \pi r^2(2r) - 2\left[\frac{1}{3}\pi r^2(r)\right] = 2\pi r^3 - \frac{2}{3}\pi r^3 = \frac{4}{3}\pi r^3$. The derivation uses Cavalieri's Principle, whereby if for two solids every plane in a set of parallel planes intersects each solid such that the two resulting cross sections have equal areas, then the solids have equal volumes. For the solid to compare with the sphere, visualize two cones as being inside a cylinder, having the same bases as the cylinder and pointing inside, sharing a common apex halfway along the cylinder. See figure (from *Geometry*, 3rd edition, by Harold R. Jacobs).




- **Tip** If you know calculus, you can determine one of the volume or surface area formulas from the other by knowing that surface area is the derivative of volume (and volume is the integral of surface area): $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 3\left(\frac{4}{3}\pi r^{3-1}\right) = 4\pi r^2$ and $\int 4\pi r^2\, dr = \frac{1}{2+1}(4\pi r^{2+1}) = \frac{4}{3}\pi r^3$. Note that the circumference and area of a circle are similarly related: $\frac{d}{dr}(\pi r^2) = 2(\pi r^{2-1}) = 2\pi r$ and $\int 2\pi r\, dr = \frac{1}{1+1}(2\pi r^{1+1}) = \pi r^2$.

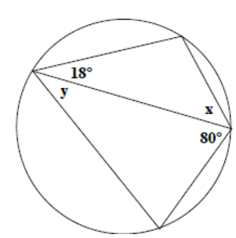
**Problems**

For the following problems, assume a calculator is not allowed unless stated.

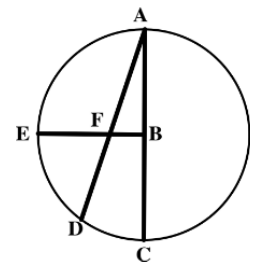
Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. A $60°$ sector of a circle has area $24\pi$. Determine exactly the radius of the circle. [calculator allowed] (MSHSML 2019-20 4B #1)
2. In the figure, a quadrilateral is inscribed in a circle. Calculate the sum of the measures of angles $x$ and $y$. [calculator allowed] (MSHSML 2018-19 4B #1)



3. In the figure, in circle $B$, $\overline{EB}$ is perpendicular to diameter $\overline{AC}$ and $m\angle EFA = 100°$. Chord $\overline{AD}$ intersects $\overline{BE}$ at $F$. What is the measure of $\widehat{DC}$? [calculator allowed] (MSHSML 2017-18 4B #1)

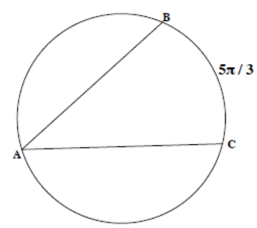



4. Determine exactly the radius of a circle in which the value of the circumference is one-third the value of its area. [calculator allowed] (MSHSML 2016-17 4B #1)

Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. One circle has a circumference of 1 and a diameter whose length is $a$. Another circle has an area of 1 and a diameter whose length is $\sqrt{b}$. Determine exactly the value of $\frac{a}{b}$. [calculator allowed] (MSHSML 2019-20 4B #2)
2. In the figure, a circle of radius 20 contains three points $A$, $B$, and $C$. Two chords, $\overline{AB}$ and $\overline{AC}$, are drawn. If the length of $\widehat{BC}$ is $\frac{5\pi}{3}$, determine exactly the measure of $\angle BAC$. [calculator allowed] (MSHSML 2018-19 4B #2)




3. In the figure, if the sum of the measures of $\widehat{AB}$ and $\widehat{DC}$ is $240°$ and $m\angle ABD = 5 \cdot m\angle A$, what is the measure of $\widehat{BC}$? [calculator allowed] (MSHSML 2017-18 4B #2)

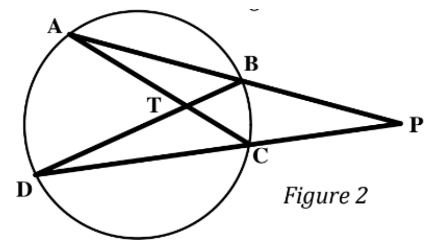
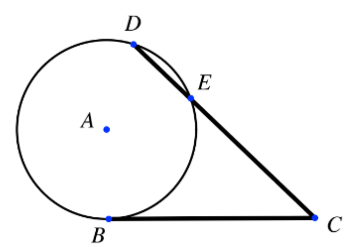



*Figure 2*

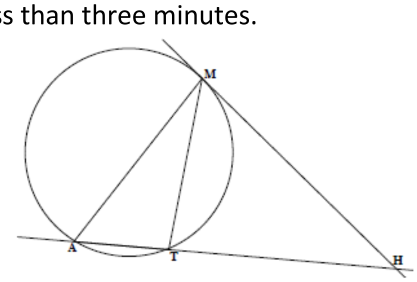
4. In the figure, secant $\overline{CD}$ and tangent $\overline{CB}$ intersect circle $A$ at points $B$ and $E$. If $BC = 10$ and $CE = 5$, determine $DE$ exactly. [calculator allowed] (MSHSML 2016-17 4B #2)



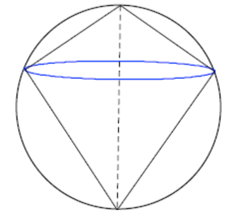

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

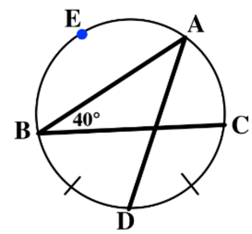
1. In the figure, $\triangle MAT$ is inscribed in a circle as shown. The tangent line from $M$ intersects $\overleftrightarrow{AT}$ at $H$. If $\widehat{AT}$ and $\angle AHM$ both have a measure of $36°$, determine exactly the measure of $\angle ATM$ in degrees. [calculator allowed] (MSHSML 2019-20 4B #3)



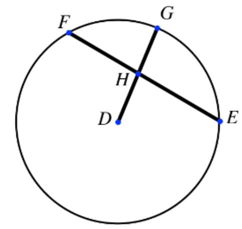
2. In the figure, a solid, formed by two right circular cones, one with a height twice the other but sharing the same base, is inscribed in a sphere of radius 12. Determine exactly the volume of the inscribed solid. [calculator allowed] (MSHSML 2018-19 4B #3)



3. In the figure, $m\angle B = 40°$, $D$ is the midpoint of $\widehat{BC}$, and the measure of each of the minor arcs $\widehat{AEB}$ and $\widehat{BDC}$ is less than $180°$. How many integer values for $m\angle A$ are possible? [calculator allowed] (MSHSML 2017-18 4B #3)



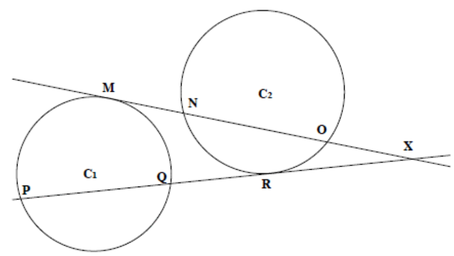

4. In circle $D$, chord $\overline{FE}$ intersects the radius $DG$ at $H$, as shown in the figure. If $FH = 3$, $HE = 5$, and $HG = 1$, determine $HD$ exactly. [calculator allowed] (MSHSML 2016-17 4B #3)



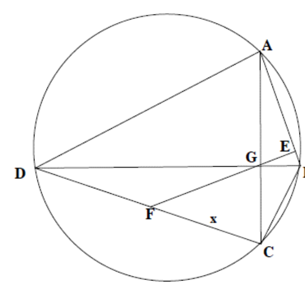

<u>Problem #4 ("challenge"; 2 points)</u>

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

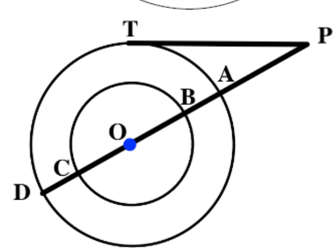
1. In the figure, $\overleftrightarrow{MX}$ is tangent to circle $C_1$ at $M$ and intersects circle $C_2$ at $N$ and $O$, and $\overleftrightarrow{PX}$ is tangent to circle $C_2$ at $R$ and intersects circle $C_1$ at $P$ and $Q$. If $MN = OX = 1$ and $PQ = RX = 2$, determine exactly $NO + QR$. [calculator allowed] (MSHSML 2019-20 4B #4)



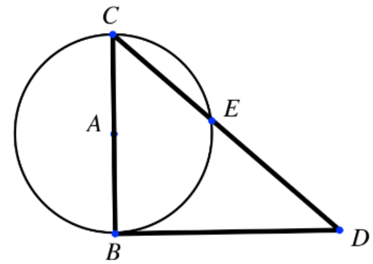

2. The figure shows a cyclic quadrilateral $ABCD$, with perpendicular diagonals, $\overline{AC}$ and $\overline{BD}$, intersecting at point $G$. From $G$ a segment is drawn perpendicular to $\overline{AB}$ at $E$ and then extended to intersect $\overline{CD}$ at $F$. If $BE = 2$, $CG = 6$, and $EG = 4$, determine exactly $CF$. [calculator allowed] (MSHSML 2018-19 4B #4)




3. The figure shows two concentric circles centered at $O$ and $\overline{PT}$ tangent to the larger circle at $T$. Center $O$ is on $\overline{PD}$ and $\overline{PD}$ intersects the larger circle at $A$ and $D$ and the smaller circle at $B$ and $C$ as shown. If $PT = 10$, $DC = 4$, and $PA$ and $OB$ are integers, what is the length of $\overline{PA}$? [calculator allowed] (MSHSML 2017-18 4B #4)



4. Triangle $CBD$ is drawn so that side $\overline{CB}$ runs through the center of circle $A$, $\overline{BD}$ is tangent to the circle at $B$, and $\overline{CD}$ intersects the circle at point $E$, as shown in the figure. If $AC = 4$ and $BD = 3$, determine $BE$ exactly. [calculator allowed] (MSHSML 2016-17 4B #4)




When you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge of other mathematical areas (algebra, for example). For more MSHSML Meet 4 Event B problems, see past exams, which date back to 1980-81.