Minnesota Junior High Math League

Practice Problems: Meet 5

Meet 5 Topics:

Numbers and Operations

- Base *n* arithmetic

Algebra

- Solving Systems of Linear Equations
- Operations with Polynomials (includes Expanding and Factoring)

Geometry and Measurement

- Similar Figures

Counting and Probability

- Probability

Data and Statistics

- Data Displays

Topic 1: Base *n* arithmetic

Numbers and Operations

1.

In base 10, what is the value of $1001_2 \times 13_8$?

Solution:

Converting both numbers to base 10, we have:

$$1001_2 \times 13_8$$
$$= (1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1) \times (1 \times 8^1 + 3)$$
$$= 9 \times 11$$
$$= 99$$

2.

The base 9 number 62_9 is equal to base b number 132_b. What is b?

Solution:

Convert to base ten:

$$62_9 = 132_b$$

$$6 \cdot 9 + 2 = b^2 + 3b + 2$$

$$54 = b^2 + 3b$$

$$54 = b(b + 3)$$

Since b is a positive integer less than 9, we can guess and check to see that $b = 6$.

3.

In base 9, the fraction $\frac{1}{7}$ is represented as the repeating decimal $0.abcabcabc\ldots_9$, where a, b, and c are digits between 0 and 8. What is the value of c?

Solution: Proceed with long division in base 9:

```
     0.1251...
7 ) 1.0000000
      7
      20
      15
       40
       38
        10
        ...
```

The key is to do all operations in base 9, e.g. $10_9 - 7_9 = 2_9$, $7_9 \times 2_9 = 15_9$, etc.

Then we see that in this base 9 representation, $c = 5$.

4.

The following equation is true when written in base n, but the first two digits of the number $\square\square 2$ are missing. What are the missing digits? Answer by filling in the boxes.

$$\square\square 2 - 135 = 234$$

Solution: Rewrite as an addition problem: $234_n + 135_n = XX2_n$.

From the units digit, since $4_n + 5_n = 2_n$ isn't true for any base n, we must have $4_n + 5_n = 12_n$. Converting to base 10 gives the equation $4 + 5 = n + 2$, so $n = 7$. Now that we know the base we can proceed as a base-7 addition problem:

```
   1
  X34
+135
  X02
```

(carry the 1, add, and the second digit is 0 since $1_7 + 3_7 + 3_7 = 10_7$)

```
 1 1
  234
+135
  402
```

(carry the 1, add, and the first digit is 4, so the answer is 402).

Topic 2: Solving Systems of Linear Equations

Algebra

5.

Anya and Adeem collect Pokemon trading cards. Together they have 95 cards, and Anya has 50%

more cards than Adeem. How many Pokemon trading cards does Adeem have?

Solution:

Suppose Anya has m Pokemon trading cards and Adeem has n cards.

Then the conditions of the question tell us that:

$$m + n = 96$$

and $m = 1.5n$

Substituting $m = 1.5n$ into $m + n = 95$:

$$1.5n + n = 95$$
$$2.5n = 95$$
$$5n = 190$$
$$n = 38$$

Therefore Adeem has 38 Pokemon cards.

6.

If (x, y) is a solution to the pair of equations

$$x + 4y = 43$$
$$4x + y = 57$$

what is the value of $x + y$?

Solution: We could proceed by substitution or elimination, but there's a nice symmetry between x and y in the equations which suggests adding both equations:

$$(x + 4y) + (4x + y) = 100$$

$$5x + 5x = 100$$

so $x + y = 20$

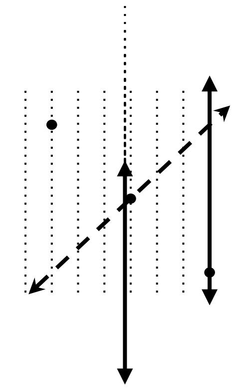
This answers the questions without even needing to find x or y!

7.

Determine the coordinates of the point P on line $y = 15 - 2x$ which is equidistant from points $(0,0)$ and $(-6,6)$ (i.e. the distance from P to $(0,0)$ is the same as the distance from P to $(-6,6)$).

Solution:

First, which points are equidistant from $(0,0)$ and $(-6,6)$? A diagram may help:



Clearly the midpoint, $(-3,3)$ is equidistant. In fact with some thought we see that any point along the dashed line shown is equidistant (whereas other points are closer to either one or the other).

This line has a y-intercept of 6 and a slope of 1, so it's equation is $y = x + 6$.

But then the point P must satisfy both equations:

$$y = 15 - 2x$$
$$y = x + 6$$

Therefore $15 - 2x = x + 6$ so $x = 3$ at the point of intersection, and substituting back $y = 9$. So P is the point $(3,9)$ which is equidistant from $(0,0)$ and $(-6,6)$ and is also on the line $y = 15 - 2x$.

Topic 3: Operations with Polynomials, including expanding and factoring

Algebra

8.

Solve for x:

$$(3x + 7)(x - 9) = 3x^2 - 108$$

Express your answer as a common fraction.

Solution:

We carefully distribute the left hand side:

$$3x(x-9)+7(x-9)=3x^2-108$$
$$3x^2-27x+7x-63=3x^2-108$$
$$-20x-63=-108$$
$$-20x=-45$$
$$x=\frac{45}{20}$$

So we conclude that $x=\frac{9}{4}$.

9.

Evaluate:

$$\frac{732^2 + 2 \times 732 \times 268 + 268^2}{387^2 - 2 \times 387 \times 137 + 137^2}$$

Solution:

Resist the urge to multiply out the expressions in the numerator and denominator: it will take a long time and almost certainly result in an error somewhere!

Instead recall that $(m+m)^2 = m^2 + 2mn + n^2$ for any m and n, so the numerator can be written as just $(732 + 268)^2 = 1000^2$.

Similarly $(m-m)^2 = m^2 - 2mn + n^2$ for any m and n, so the denominator equals simply $(387 - 137)^2 = 250^2$.

Putting these together, our original fraction evaluates to:

$$\frac{1000^2}{250^2} = \left(\frac{1000}{250}\right)^2 = 4^2 = 16$$

So this expression has a value of *16*.

10.

Find the largest possible value of r such that $(r-9)(r+9) = 8r + 47$.

Solution:

Distribute the left hand side (or recognize the difference of squares identity):

$$r(r+9) - 9(r+9) = 8r + 47$$
$$r^2 + 9r - 9r - 81 = 8r + 47$$
$$r^2 - 81 = 8r + 47$$
$$r^2 - 8r - 81 - 47 = 0$$
$$r^2 - 8r - 128 = 0$$

This factors as

$$(r-16)(r-8) = 0$$

So $r = 16$ or $r = 8$ are the two solutions. Then the largest value of r satisfying the original equation is *16*.

11.

Find the sum of the solutions to the equation $x^2 - 6x + 9 = 0$

Solution:

This factors as $(x-3)(x-3)=0$ (or $(x-3)^2=0$), so $x=3$ is the only solution, and the sum of the solutions to this equation is 3.

12.

Evaluate:

$$\sqrt{12+\sqrt{12+\sqrt{12+\ldots}}}$$

Solution:Not having a better idea let's assign a variable:

$$x = \sqrt{12 + \sqrt{12 + \sqrt{12 + \ldots}}}$$

Then

$$x = \sqrt{12 + \left(\sqrt{12 + \sqrt{12 + \ldots}}\right)}$$

Note the expression in parentheses is also just x, so we have the equation

$$x = \sqrt{12 + x}$$

Squaring both sides gives

$$x^2 = 12 + x$$

Or, in standard form

$$x^2 - x - 12 = 0$$

Factoring this gives

$$(x-4)(x+3)=0$$

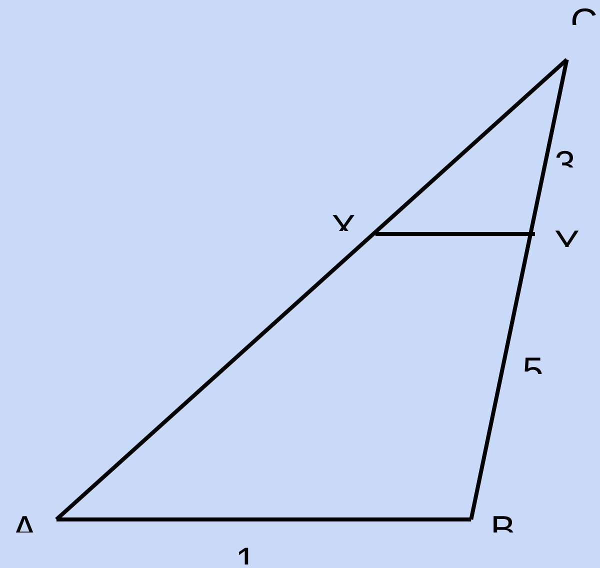
So either $(x-4)=0$ or $(x+3)=0$. The latter doesn't make sense since clearly $x>0$, so we must have $(x-4)=0$, in other words $x=4$ so the value of this infinite expression must be 4.

Topic 4: Similar Figures

Geometry and Measurement

13.

In the figure shown, XY and AB are parallel. If |AB| = 10, |BY| = 5, and |CY| = 3, determine |XY|. Express your answer as a common fraction.



Solution:

Notice that triangles XYC and ABC are similar since all three corresponding angles are equal, so therefore their sides share a common ratio.

In particular $|XY|:|YC| = |AB|:|BC|$

Substituting known side lengths gives

$$|XY|:3 = 10:8$$

or $\frac{|XY|}{3} = \frac{10}{8}$

$$|XY| = \frac{3\times 10}{8}$$

$$|XY| = \frac{15}{4}$$

14.

An Olympic sized swimming pool is 50m long and filled with 2,500,000 liters of water. Putri's backyard pool has the exact same proportions but is only 10m long. How many liters of water are needed to fill Putri's pool?

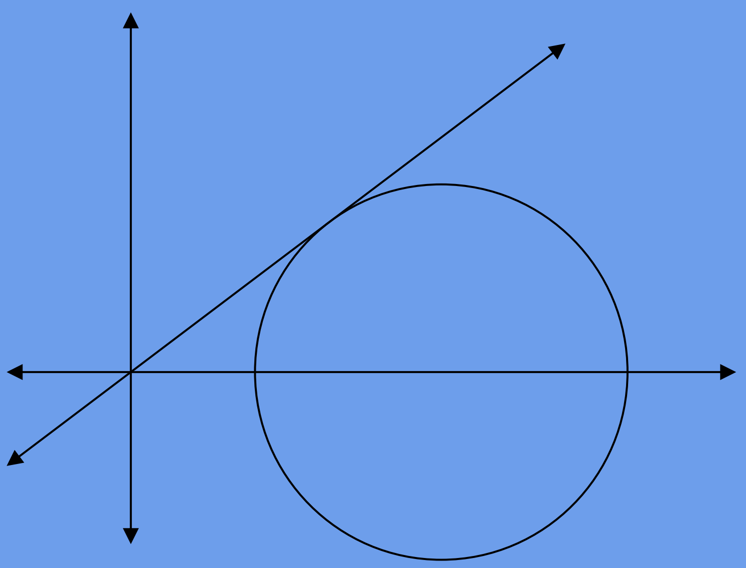
Solution:

Since her backyard pool was built using the same proportions as the original and its length is $\frac{1}{5}$ as long, it must also be $\frac{1}{5}$ as wide and $\frac{1}{5}$ as deep. This means that it's volume is $\frac{1}{5} \times \frac{1}{5} \times \frac{1}{5}$ or $\frac{1}{125}$ as much as the original.

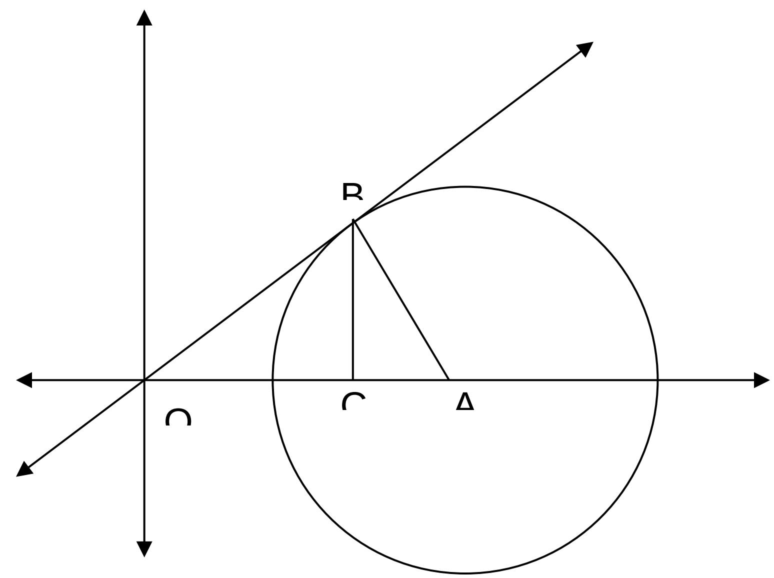
Since an Olympic sized pool contains 2,500,000 liters of water, Putri's pool requires $\frac{1}{125} \times 2{,}500{,}000 = 20{,}000$ liters of water.

15.

Determine the equation of the line shown which passes through the origin and Quadrants I and III and is tangent to the circle centered at (5, 0) with radius 3.



Solution:



We label some points and draw some additional lines: label O at the origin, A(0,5) as the circle's center, and B as the point where the line is tangent to the circle. Draw in segment AB and draw a perpendicular from B to OA at C.

Now, |OA|=5 and |AB|=3, so since OAB is a right triangle by the Pythagorean Theorem (or recognizing a special right triangle) we can see that OB=4.

To find the equation of our line we'd like to know its slope, which is $\frac{|BC|}{|OC|}$.

But notice that triangles OCB and OBA are similar (both share a common angle and have a right angle), so we must have:

$$\frac{|BC|}{|OC|} = \frac{|AB|}{|OB|} = \frac{3}{4}$$

Then we see that our line has a slope of $\frac{3}{4}$ and a y-intercept of 0, so its equation is

$$y = \frac{3}{4}x$$

Topic 5: Probability

Counting and Probability

16.

A sock drawer contains 10 black socks, 6 white socks, and 4 blue socks. Hamza reaches in and randomly pulls out two socks. What is the probability that both are black? Express your answer as a common fraction.

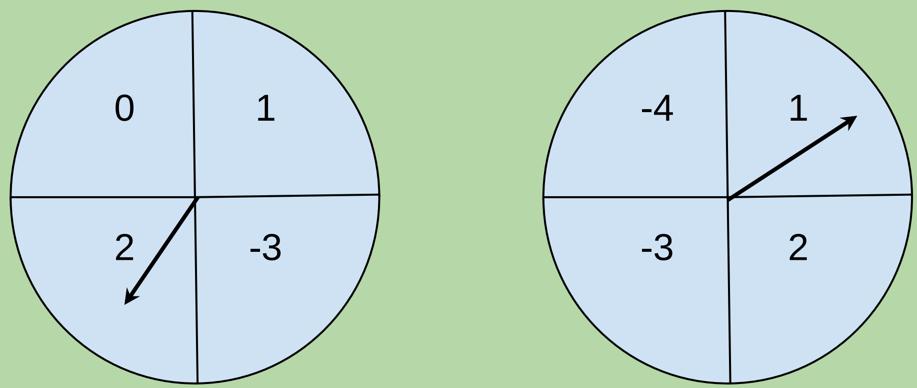
Solution:

10 of the 20 socks in the drawer are black, so the probability that the first sock drawn is black will be $\frac{1}{2}$. Once this sock has been drawn, 9 of the 19 remaining socks are black, so the probability that the second sock drawn is black is $\frac{9}{19}$.

By the multiplication principle, the probability that both socks drawn are black is $\frac{1}{2} \times \frac{9}{19} = \frac{9}{38}$.

17.

The two spinners below are divided into four regions each with an equal likelihood of being selected. Both spinners are spun and the two numbers produced are multiplied. What is the probability that this product is positive?



Solution:

There are two possible ways to get a positive product: either both numbers spun are positive, or both numbers spun are negative. Since they're exclusive events we can add the two probabilities.

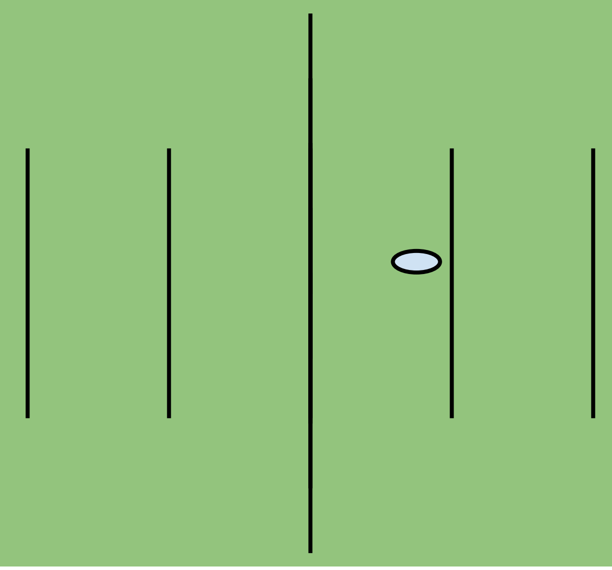
The probability that both numbers spun are positive is $\frac{2}{4} \cdot \frac{2}{4} = \frac{1}{4}$ (since there are two positive numbers on each spinner).

The probability that both numbers spun are negative is $\frac{1}{4} \cdot \frac{2}{4} = \frac{1}{8}$ (one negative number on the first spinner, two on the second).

Overall, this gives a $\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$ probability that the product of the numbers spun is positive.

18.

A quarter of diameter 1” is tossed onto a square grid with rows and columns exactly 3” apart. What is the probability that it lands completely within one square, not touching any of the horizontal or vertical lines, as in the example shown? Express your answer as a common fraction.



Solution: Consider the center of the quarter. If it lands more than 0.5” from any line, then the quarter isn’t touching any of the lines (and otherwise it is). In other words, if the center of the quarter is within the central 2”×2” square inside each 3”×3” grid square then it won’t be touching a line.

Then since the quarter lands randomly within some square, the probability that it doesn’t touch any line is the same as the ratio of the area of the inner 2”×2” square to the area of the full 3”×3” square, which is $\frac{4}{9}$.

19.

Four players on the soccer team left their athletic bags on the sideline during practice. If each one grabs a bag at random when they leave, what is the probability that no player gets the correct bag? Express your answer as a common fraction.

Solution:In total, there are 4! = 24 ways to randomly assign the four bags to the four players (four choices for the first player, then 3 for the second, 2 for the third, and the fourth player gets the remaining bag).

For notational purposes, we'll designate an assignment by a four digit number where the first digit indicates which bag the first player got, the second shows which bag the second player got, and so on.

For instance 1432 would indicate that players 1 and 3 got their own bags, but players 2 and 4 got each-other's bag.

Using this notation we can carefully list assignments in which no player gets their own bag. We list these in numerical order to make sure we don't miss any:

2143   2341   2413   3142   3412   3421   4123   4312   4321

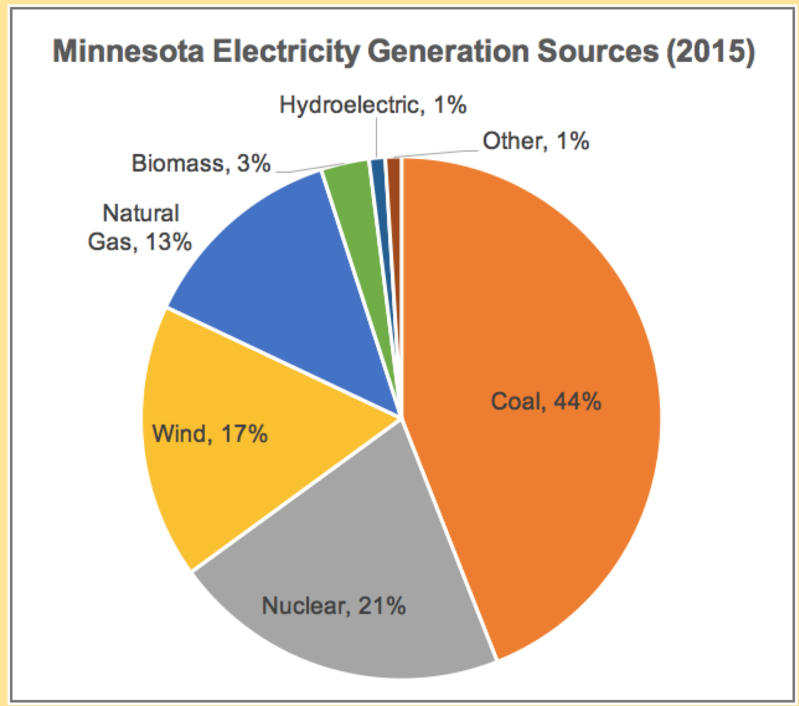
This is 9 arrangements in which no player got their own bag, so the probability that no player gets the correct bag is $\frac{9}{24}$ or $\frac{3}{8}$.

Topic 6: Data Displays

Data and Statistics

20.

Coal and Natural Gas are considered fossil fuels. Based on the chart given, what percentage of Minnesota's 2015 electricity was generated from sources other than fossil fuels?



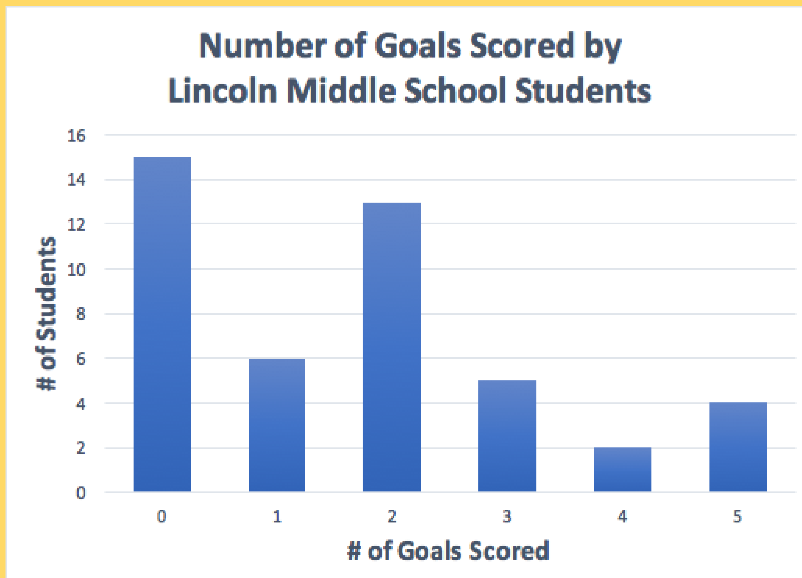
Solution:

Since 44% of electricity was generated from Coal and 13% from Natural Gas, in total 57% of electricity was generated from fossil fuels.

Then the remaining 100%-57% or 43% of electricity was generated from sources other than fossil fuels.

21.

45 students at Lincoln Middle School played soccer last season. Based on the frequency histogram shown, by how many goals is the median number of goals scored per student greater than the mean number of goals scored per student? Express your answer as a common fraction.



Solution: Reading the graph we see that 15 students scored 0 goals, 6 scores 1 goal, 13 scored 2 goals, 5 scored 3 goals, 2 scored 4 goals, and 4 scored 5 goals.

In order by goals scored, the 23rd student scored 2 goals, so the median number of goals scored per student is 2.

To find the mean, we add up the number of goals scored by all students. We'll have 15 students adding 0 goals each, 6 adding 1 goal each, and so on, so the total number of goals scored by all 45 students is

$$15 \times 0 + 6 \times 1 + 13 \times 2 + 5 \times 3 + 2 \times 4 + 4 \times 5$$
$$= 0 + 6 + 26 + 15 + 8 + 20$$
$$= 75$$

Then the average number of goals scored was $\frac{75}{45} = \frac{5}{3}$.

Now we can see that the median number of goals scored (2) is $\frac{1}{3}$more than the mean number of goals scores ($\frac{5}{3}$), so the answer is $\frac{1}{3}$.

22.

Lake County, Minnesota has a population of 10,800, broken into age categories as indicated in the table shown (right).

Age	Population
0 - 9	1,200
10 - 19	1,500
20 - 29	900
30 - 39	1,300
40 - 49	1,800
50 - 59	1,400
60 - 69	1,200
70 - 79	1,000
80+	500

Logan wants to present this information in a pie chart. How many degrees should he measure for the central angle of the "Age 10-19" sector?

Solution: There are 360° in a circle, so with a total population of 10,800 every degree in the circle will represent $\frac{10800}{360} = 30$ people.

Then with 1,500 people in Lake County in the 10-19 age category, this segment of the pie chart should have a central angle of $\frac{1500}{30} = 50$ degrees.