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Practice Problems

MNJHML Meet 3 Answer Key

n

1. 15,300
2. N=238
3. 3:36 PM

   1
4. 40%
5. 297 square feet
6. 3 fidget spinners
7. 51%
8. $= \frac{2}{7}$
9. 1
10. $= \frac{x}{6}$
11. n=-10
12. $= 4\sqrt{3}$
13. 6
14. .707
15. 7 integers
16. x=0
17. 36 minutes
18. 18 years old
19. 28%
20. 644 square inches
21. 24cm
22. 5cm
23. 45 combinations
24. 10 routes
25. 56 triangles
26. 21 ways to arrange books.

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Meet 3 Practice Problems

Complete Solutions

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1. The least common multiple of 1, 2, 3, 4 and 5 is 60. This means that since 15,240 is divisible by 60, the next larger integer also divisible by 60 is 15,240 + 60 = 15,300.
2. Since 34 is a factor of *N*, and since *N* is between 200 and 300, the only three options for *N* are 6×34=204, 7×34=238, and 8×34=272.
   Now, 748=22×34; in particular 748 is divisible by 68 (2×34). Since GCF(748, *N*) = 34, *N* can't also be divisible by 68. This leaves only one possible value for *N*, namely 238.
3. The bells ring together at every common multiple of 72 and 54 minutes past noon. Since LCM(72, 54) = LCM(4×18, 3×18) = 12×18 = 216, they will ring together for the first time exactly 216 minutes past noon.

   Now, 3 hours is 180 minutes, so 216 minutes is 3 hours and 36 minutes. This means that the bells will next ring together at 3:36 PM.
4. We factor $m = 2^2 \cdot 3$, $n = 2^2 \cdot 7$. Then $LCM(m,n) = 2^2 \cdot 3 \cdot 7$ and $GCF(m,n) = 2^2$.

   Then we have $$\frac{LCM(m,n)\cdot GCF(m,n)}{m\cdot n} = \frac{(2^2\cdot 3\cdot 7)\cdot(2^2)}{(2^2\cdot 3)(2^2\cdot 7)} = \frac{2^4\cdot 3\cdot 7}{2^4\cdot 3\cdot 7} = 1$$

   In fact $\frac{LCM(m,n)\cdot GCF(m,n)}{m\cdot n} = 1$ for all values of m and n; can you see why?
5. Since 14 of the 35 students are boys, the fraction of boys in the club is
   $$\frac{14}{35} = \frac{2\times 7}{5\times 7} = \frac{2}{5}$$
   We want a denominator of 100 to convert this to a percent:
   $$\frac{2}{5} = \frac{2\times 20}{5\times 20} = \frac{40}{100}$$
   So we conclude that 40% of the club members are boys.
6. This year his garden is
   25×1.1 feet long (10% longer than last year), and 12×0.9 feet wide (10% shorter than last year).

   Its area this year is therefore:

   $$(25\times 1.1)\times(12\times 0.9) = (25\times 12)\times(1.1\times 0.9) = 300\times 0.99 = 297 \text{ square feet}$$

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(Note that we could have determined the dimensions as 27.5 and 10.8 feet and multiplied these; by holding off on multiplying until the end we're able to rearrange the terms and make the arithmetic simpler)

7. Suppose she started with x fidget spinners. Then she gave $0.2x$ spinners to Vance, keeping the remaining $0.8x$. She gave 50% of these to Jill, which means that Jill got $0.4x$ spinners and Keiko kept $0.4x$ spinners for herself. We're told that she ended with 6 spinners, which means:

$$0.4x = 6$$
$$4x = 60$$
$$x = 15$$

So she started with 15 fidget spinners. Since she gave 20% of these to Vance, he got 3 fidget spinners.

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8. At the end of the first year, 80% of his hair remains (or 0.8 as a decimal)
He keeps 80% of this through his second year, ending with 0.8 × 0.8 = 0.64 or 64% of the hair he started with.
At the end of his third year, he'll still have 80% of this, which is 0.8 × 0.64 = 0.512 or 51.2%. So after three years he'll still have 51% of his hair, to the nearest percent.

9. $$\frac{2\times 3^{-1}}{2+3^{-1}} = \frac{2\times\frac{1}{3}}{2+\frac{1}{3}}$$
$$= \frac{3(2\times\frac{1}{3})}{3(2+\frac{1}{3})}$$
$$= \frac{2}{6+1}$$
$$= \frac{2}{7}$$

10. The base of this is going to be big; let's look at the exponent first:

$$m^2 - 8m + 7 = 7^2 - 8\cdot 7 + 7$$
$$= 49 - 56 + 7$$
$$= 0$$

Perfect! Any nonzero number raised to the power 0 is 1, and since the base of this exponent is clearly a large number, we see that $(m^3 + 9m^2 + 11m + 5)^{m^2-8m+7} = 1$ when $m = 7$.

11. $$\frac{(3x^2)^{-1}}{2x^{-3}} = \frac{\frac{1}{(3x^2)}}{\frac{2}{x^3}}$$

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$$= \frac{\frac{1}{(3x^2)} \cdot 3x^3}{\frac{2}{x^3} \cdot 3x^3}$$
$$= \frac{x}{6}$$

12. We could actually evaluate each of the exponents in the denominator of this fraction, but let's try pulling out a common factor of 2^{10} first to see where that gets us:

$$\frac{1}{2^{12}-2^{11}-2^{10}} = \frac{1}{2^{10}(2^2-2^1-1)}$$
$$= \frac{1}{2^{10}(4-2-1)}$$
$$= \frac{1}{2^{10}}$$
$$= 2^{-10}$$

This means that $n = -10$.

13. $\sqrt{8^2 - 4^2} = \sqrt{64 - 16}$
$= \sqrt{48}$
$= \sqrt{4^2 \cdot 3}$
$= 4\sqrt{3}$

14. $\frac{\sqrt{108}}{\sqrt{3}} = \sqrt{\frac{108}{3}}$
$= \sqrt{36}$
$=6$

15. $\sqrt{0.5} = \sqrt{\frac{1}{2}}$
$= \frac{1}{\sqrt{2}}$
$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{\sqrt{2}}{2}$

Since $\sqrt{2}$ is 1.414 to the nearest thousandth, $\frac{\sqrt{2}}{2}$ is 0.707 to the nearest thousandth, so $\sqrt{0.5}$ is also 0.707 to the nearest thousandth.

16. Since $\left(\frac{8}{3}\right)^2 = \frac{64}{9}$ which is between 7 and 8, $\sqrt{7} < \frac{8}{3} < \sqrt{8}$.

Since $\left(\frac{15}{4}\right)^2 = \frac{225}{16}$ is between 14 and 15, $\sqrt{14} < \frac{15}{4} < \sqrt{15}$.

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Thus we can see that each of the integers 8, 9, 10, 11, 12, 13, and 14 has a square root between $\frac{8}{3}$ and $\frac{15}{4}$, which is a total of 7 such integers.

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17. Multiply both sides by $1 + \sqrt{x}$ to clear fractions:

$$5 + 3(1 + \sqrt{x}) = 8$$
$$8 + 3\sqrt{x} = 8$$
$$\sqrt{x} = 0$$

But this is only possible when $x = 0$ which we can verify is a solution to the original equation, so this is the only solution.

18. Ralph clears $\frac{1}{45}$ of the driveway every minute. If Sheila can clear the driveway in s minutes by herself, then she can clear $\frac{1}{s}$ of the driveway every minute.

Working together they can clear the entire driveway in 20 minutes, so in one minute they clear $\frac{1}{20}$ of the driveway.

This leads to the equation:

$$\frac{1}{45} + \frac{1}{s} = \frac{1}{20}$$

so
$$\frac{1}{s} = \frac{1}{20} - \frac{1}{45}$$
$$\frac{1}{s} = \frac{9}{180} - \frac{4}{180}$$
$$\frac{1}{s} = \frac{5}{180}$$
$$\frac{1}{s} = \frac{1}{36}$$

Which means that $s = 36$. So Shiela can clear the driveway in 36 minutes when she works alone.

19. Following the pattern, when Retu turns n she gets to eat $\frac{n}{9+n}$ of the cake. Since she's eating $\frac{2}{3}$ of her cake today, $\frac{n}{9+n} = \frac{2}{3}$

so
$$3n = 2(9 + n)$$
$$3n = 18 + 2n$$
$$n = 18$$

Therefore today was Retu's 18th birthday and she's now 18 years old.

20. Suppose the original cylinder has height h and radius r. Then its volume is $\pi r^2 h$.

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When the height is increased by 100%, the new height is $(1+\frac{100}{100})h = 2h$. When the radius is decreased by 20%, the new radius is $(1-\frac{20}{100})r = 0.8r$. Then the new volume is $\pi(0.8r)^2(2h) = \pi \cdot 0.64 \cdot r^2 \cdot 2 \cdot h = 1.28\pi r^2 h$.

This is exactly a factor of 1.28 more than the original volume, so the net result is an increase in volume of 28%.

21. The box's exterior bottom and two larger sides each have an area of 8×10=80 square inches. Its two smaller exterior ends each have an area of 8×8=64 square inches. Looking from directly above, the top surfaces of the upper lip plus the inside bottom of the crate combined have a surface area of 8×10=80 square inches (alternately the upper lip and inside bottom areas can be computed separately). The inner smaller ends are each 7×6=42 square inches, and the two inner sides each have an area of 7×8=56 square inches.

    This accounts for the entire surface of the crate, so its total surface area is 3×80+2×64+80+2×42+2×56 = 644 square inches.

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22. Suppose the bucket has an inner radius of r and an initial water depth of h. Then the volume of water in the bucket is $\pi r^2 h$.

    After the rock is added, the water level rises 0.5 cm so the new volume of water is $\pi r^2(h+0.5)$. This is an increase in volume of $\pi r^2(0.5) = \frac{1}{2}\pi r^2$, which must be the volume of the rock which was added.

    A spherical rock of diameter 12 cm has a radius of 6 cm, hence a volume of $\frac{4}{3}\pi(6)^3 = 288\pi$. Equating these two expressions for the rock volume gives:

    $$\frac{1}{2}\pi r^2 = 288\pi$$

    so $\quad r^2 = 576 = 4 \times 144$

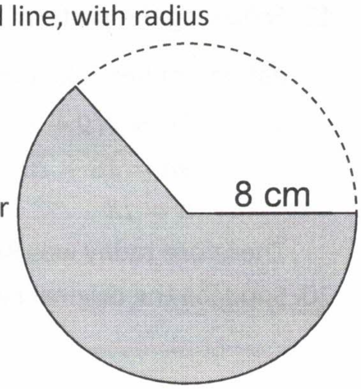
    and $\quad r = 2 \times 12 = 24$

    This means that the inner radius of the bucket must have been 24 cm.

23. Consider the area of the whole circle shown by the dashed line, with radius $R = 8$. This is $\pi R^2 = 64\pi$.

    So the ratio of the sector to the area of the whole circle is $\frac{40\pi}{64\pi} = \frac{5}{8}$.

    That means the length of the circular arc around the sector is ⅝ of the circumference of the whole circle, so the arc length of the sector is $\frac{5}{8} \cdot 2\pi R = 10\pi$.



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But the arc length of the sector is equal to the circumference C of the circular base of the cone when it is curled into shape--therefore, the radius r of the base is

$$r = \frac{C}{2\pi} = \frac{10\pi}{2\pi} = 5 \text{ cm}$$

24. Using combinations, the number of ways to select two donuts of different types from among 10 varieties is

$$\binom{10}{2} = \frac{10!}{2!\,8!} = \frac{10\times 9}{1\times 2} = 45.$$

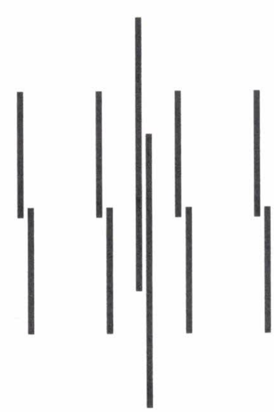
Alternately there are 10 choices for the first donut selected and 9 for the second for a total of 90 possibilities. However this counts each possibility twice (picking maple glaze first and double chocolate second leads to the same snack as picking double chocolate first and maple glaze second), so the true number of snacks possible is 90÷2=45 as before.

25. *Solution 1:*

A valid route is a sequence of five segments, each of which is either “north” or “east”, which must include two “north” segments and three “east” segments. The number of paths is equal to the number of ways of selecting the two “north” segments within the five block route, which is $\binom{5}{2} = \frac{5!}{2!3!} = \frac{5\times 4}{1\times 2} = 10.$

Solution 2:

Alternately, path counting problems such as this can be tackled algorithmically. Suppose the intersections were labelled A through L as shown at right; we determine the number of paths from A to each intersection starting with those close to A until we’ve counted all paths to point L (the park).



- There’s only a single way to get from A to either B, C, D, F, or I, so we label those intersections with this count (1)
- A trip to point E must have come through either B (one way to get there) or C (one way to get there), for a total of two paths, so we label this with a 2
- Point G must have been reached from either D (1 way) or E (2 ways), so there are 3 ways to get to point G

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- Point H must have been reached from E (2 ways) or F (1 way): 3 total
- Point J must have been reached from G (3 ways) or H (3 ways): 6 total
- Point K must have been reached from H (3 ways) or I (1 way): 4 total
- Point L must have been reached from J (6 ways) or K (4 ways): 10 total

It's now clear that there are exactly 10 paths that Maya could have taken.

26. Solution: A triangle is uniquely defined by its three vertices, which are in turn selected from among the eight vertices of the octagon. There are $\binom{8}{3} = \frac{8!}{3!\,5!} = \frac{8\cdot7\cdot6}{1\cdot2\cdot3} = 56$ possible ways to do this, so there are 56 such triangles. $\binom{8}{3}$

27. Line up the 8 identical books and consider the 7 "gaps" between books. Selecting two of these 7 gaps splits the books into three groups; we can put the leftmost group on the top shelf, the middle on the middle shelf, and the rightmost group on the bottom shelf. Since these arrangements are unique and cover all possible arrangements, there are be $\binom{7}{2} = 21$ ways to arrange the books.

$$\binom{7}{2} = \frac{7!}{2!(7-2)!} = \frac{7!}{2!\,5!} = \frac{7\cdot6\cdot\cancel{5!}}{2\cdot1\cdot\cancel{5!}} = \frac{42}{2} = \boxed{21}$$