Meet 4 Answers

1. 0
2. 2
3. 1,185,921
4. 2
5. 9
6. s=-9
7. 13 integers
8. 209 puzzles
9. 15
10. $\frac{7}{2}$
11. 26
12. $\frac{25}{3}$
13. 20
14. 162 new cases
15. 5 triangles
16. 41 dots
17. A=14
18. |AB| = 16cm
19. $4\sqrt{3}$ cubic feet
20. 36 square units
21. 5 ways to arrange the numbers
22. 26 paths
23. 44 right triangles
24. 1234567
25. 20 points per game
26. Range is 7
27. x=30

Meet 4 Complete Solutions

1. Let's look at last digits of powers of 9:

   - $9^1 = 9$ ends in 9
   - $9^2 = 81$ ends in 1
   - $9^3 = 9^2 \times 9$ ends in the same digit as 1×9, i.e. 9
   - $9^4 = 9^3 \times 9$ ends in the same digit as 9×9, i.e. 1

   So we see that odd powers of 9 end in 9, and even powers of 9 end in 1.

   Then 9^4+9^7 ends with the same digit as $1 + 9 = 10$, so $9^4 + 9^7$ has a last digit of 0.
2. If any of the three numbers were divisible by 4, their product would be also, so there would be no remainder when this is divided by 4. So the only possible way for three consecutive numbers to leave a remainder when multiplied is when they're of the form $4m + 1$, $4m + 2$, $4m + 3$ for some integer m.

   Now, $(4m + 1)(4m + 2)(4m + 3)$ leaves the same remainder when divided by 4 as $1 \times 2 \times 3 = 6$ does, since any multiples of 4 are discarded. Since 6 leaves a remainder of 2 when divided by 4, we conclude that the three numbers in question must also leave a remainder of 2 when divided by 4.

   Of course we could just pick the three consecutive numbers 1, 2, 3, multiply them to get 6, and see that the remainder of these two numbers on division by 4 is 2. The question implies that any nonzero number will always leave the same remainder, so with that assumption we know right away that the answer must be 2.
3. These are big numbers, don't seem easy to factor, and we don't have a calculator to work with. There's got to be some sort of trick, and since the units digit is a topic this meet let's think about the units digit of perfect squares.

   We know that 1^2 ends in 1, 2^2 ends in 4, 3^2 ends in 9, 4^2 ends in 6, 5^2 ends in 5, 6^2 ends in 6, 7^2 ends in 9, 8^2 ends in 4, 9^2 ends in 1, and 10^2 ends in 0.

   What about larger numbers? When squared, their units digit is the same as the units digit of the number squares, do must also be one of the above number.

   So perfect squares must end in 0, 1, 4, 5, 6, 9.

Aha! Of the numbers listed, all but one ends in some other digit - so since we're told that one of them is a perfect square, is must be 1,185,921

4. First, notice that since $3^4 = 81$, $2^{3^4} = 2^{81}$

   What is the units digit of 2^{81}? Let's again start with small cases:

   - 2^1 ends with 2
   - 2^2 ends with 4
   - 2^3 ends with 8
   - 2^4 ends with 6
   - 2^5 ends with 2
   - 2^6 ends with 4
   - … and so on ...

   This pattern repeats since only the last digit of the previous power matters in determining the units digit of the next higher power.

   So we see that every power of 2 which is a multiple of 4 ends in a 6, and in particular 2^{80} ends in a 6, which means that 2^{81} has a units digit of 2.
5. Using long division we can see that 2018 ÷ 13 is 155 with a remainder of 3.
   Squaring 3 more than a multiple of 13 leaves the same remainder on division by 13 as just squaring 3, which is 9, so we conclude that 2018^2 leaves a remainder of 9 when divided by 13.
6. Subtract $6s$ from both sides: $-5 > 3s + 19$
   Subtract 13: $-24 > 3s$
   Divide by 3 $-8 > s$
   Which is equivalent to $s < -8$.

   As a result, the greatest integer value of s satisfying this inequality is $s = -9$.
7. First we solve: $2n - 7 > 0$
   Add 7: $2n > 7$
   Divide by 2: $n > \frac{7}{2}$

   Next we solve: $50 - 3n > 0$
   Subtract 50: $-3n > -50$
   Divide by -3: $n < \frac{50}{3}$ (remember to reverse the inequality)
   Since n is an integer, the first inequality tells us that $n \geq 4$ and the second that $n \leq 16$Thus any of the integers 4, 5, 6, …, 15, 16 satisfy the inequality: 16 - 4 + 1 = 13

numbers in all. So we see that there are 13 integers n for which both of these expressions are positive.

8. Galois Games sells n puzzles for $500 + 3.5n$, and Noether Novelties charges $750 + 2.3n$ for the same order.
   So Noether Novelties offers a better deal whenever
$$750 + 2.3n < 500 + 3.5n$$
$$250 < 1.2n$$
$$n > \frac{250}{1.2}$$
   Using long division we see that $\frac{250}{1.2}$ is between 208 and 209, so Noether Novelties offers a lower total manufacturing cost for orders of 209 puzzles or more.

9. This compound inequality is equivalent to solving both

$-x \leq 3$	and	$3 < 10 - x$
$x \geq -3$		$x + 3 < 10$
		$x < 7$

   Both inequalities are true for the integers -3, -2, -1, …, 4, 5, 6. Adding these up:
$$\begin{aligned}&(-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6\\ &= (-3 + 3) + (-2 + 2) + (-1 + 1) + 0 + 4 + 5 + 6\\ &= 4 + 5 + 6\\ &= 15\end{aligned}$$

   So the sum of all integer solutions to this inequality is 15.

10. Moving from (2, -1) to (8, 3) involves moving +6 in the x direction and +4 in the y direction. So along this line, moving +1 in the y direction leads to an increase of $+\frac{6}{4}$ or 1.5 in the x direction.
    Using this, from (2, -1) we could move to (3.5, 0) along the line (adding 1 to y and 1.5 to x), so the x-intercept of this line is 3.5 or $\frac{7}{2}$ as a common fraction.

11. Suppose there are n members. Then they contributed $8n$ dollars to cover the cost of shirts, which from the cost equation must have cost $13 + 7.5n$ dollars. Since contributions exactly covered the cost of shirts,
$$\begin{aligned}8n &= 13 + 7.5n\\ 0.5n &= 13\\ n &= 26\end{aligned}$$

    So 26 shirts were purchased, and there are 26 members in the math club.

12. The x-intercept of the line $3x + 8y = 20$ occurs when $y = 0$:

$$3x + 8(0) = 20 \qquad \Rightarrow \qquad 3x = 20 \qquad \Rightarrow \qquad x = \frac{20}{3}$$

The y-intercept of this line occurs when $x = 0$:

$$3(0) + 8y = 20 \qquad \Rightarrow \qquad 8y = 20 \qquad \Rightarrow \qquad y = \frac{5}{2}$$

Now we see that the triangle in question has base length $\frac{20}{3}$, height $\frac{5}{2}$, so its area is:

$$\frac{1}{2}\left(\frac{20}{3}\right)\left(\frac{5}{2}\right) = \left(\frac{10}{3}\right)\left(\frac{5}{2}\right) = \left(\frac{5}{3}\right)(5) = \frac{25}{3}$$

13. The 14th term of the sequence is 21 more than its 7th term, and moving from the 7th to the 14th term involves adding seven common differences, so we must have a common difference of $\frac{21}{7}$ or 3.
    The 10th term is reached by adding three common differences to the 7th term, so it must be 11+3(3)=20. Thus the 10th term in this sequence is 20.
14. Suppose there were b cases of bird flu 3 months ago. Then there were $\frac{2}{3}b$ new cases 2 months ago, $\left(\frac{2}{3}\right)^2 b$ last month, and $\left(\frac{2}{3}\right)^3 b$ this month. Since there were 48 new cases this month:

$$\left(\frac{2}{3}\right)^3 b = 48 \qquad \Rightarrow \qquad \frac{8}{27}b = 48 \qquad \Rightarrow \qquad \frac{1}{27}b = 6 \qquad \Rightarrow \qquad b = 162$$

    This means there were 162 new cases of bird flu 3 months ago.
15. We classify all arithmetic sequences of positive integers at most 7 by their common difference. The common difference can't be zero since the triangle isn't equilateral, and negative common differences gives the same sequence in reverse (hence the same triangle)

    Common difference 1:
    1, 2, 3   2, 3, 4   3, 4, 5   4, 5, 6   5, 6, 7

    Common difference 2:
    1, 3, 5   2, 4, 6   3, 5, 7

    Common difference 3:
    1, 4, 7

Not all of these nine arithmetic sequences can form a triangle, however, since the sum of the length of the two smaller sides must be greater than the length of the longest side.

This leaves the following valid sequences which can form a triangle:

2, 3, 4   3, 4, 5   4, 5, 6   5, 6, 7   3, 5, 7

So there are 5 such triangles.

16. The first stage has 1 dot, the second has 5, the third 9, and so on with 4 more dots added at each stage (one at the top, one at the bottom, and two to the right).

These numbers form an arithmetic sequence with an initial term of 1 and common difference of 4, so the 11th term in this sequence is $1 + 4 \times (11 - 1) \;=\; 41$. So the 11th stage in this pattern will include 41 dots.

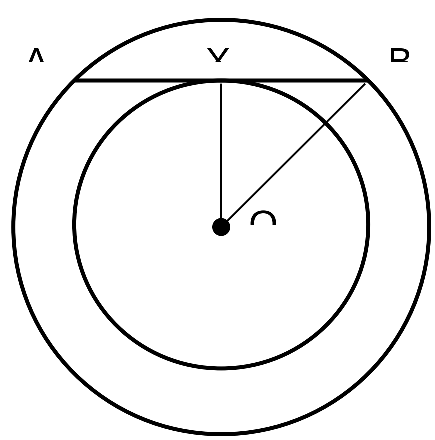
17. Let the other leg of this triangle have length b. Then by the Pythagorean Theorem:

$$b^2 + 4^2 = \left(\sqrt{65}\right)^2 \quad \Rightarrow \quad b^2 + 16 = 65 \;\Rightarrow\quad b^2 = 49$$

So $b = 7$ (since it must be positive). So the area of this triangle is just $\frac{1}{2}(4)(7) = 14$.

18. Label the tangent point of AB to the smaller circle as X, and draw in lines OX and OB as shown.

Then |OX| = 6 cm and |OB| = 10 cm since these are radii of the smaller and larger circles, respectively. Since X is tangent to the smaller circle, OXB is a right-angled triangle, so we can apply the Pythagorean theorem:



$|OX|^2 + |XB|^2 = |OB|^2$,

so $\quad 6^2 + |XB|^2 = 10^2$

$36 + |XB|^2 = 100$

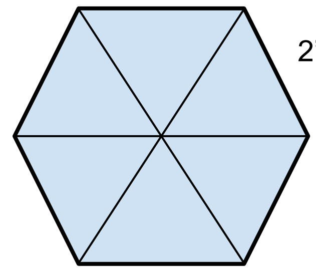
$|XB|^2 = 64$

$|XB| \;\; = 8$ (since it's positive)

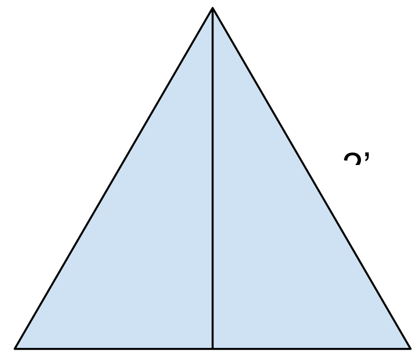
But X is the midpoint of AB, so we must have |AB| = 16 cm.

(Note that we could also recognize XOB as a special 3-4-5 triangle with side lengths multiplied by 2 to immediately see that |XB|=8)

19. Notice that the hexagonal shape can be split into six equilateral triangles as shown at right. To find the pool’s volume we first need to know the area of its base; to do this we’ll find the area of an equilateral triangle with side length 2 feet.



Consider one such triangle enlarged to the right: we can drop an altitude which will split the base in the middle.Then either using the Pythagorean Theorem or by recognizing a special 30-60-90 triangle, we determine that this altitude has length $\sqrt{3}$ feet, so the triangle’s area is $\frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$ square feet.



Then the area of the base of the pool must be $6\sqrt{3}$ square feet, and the pool is filled to a depth of 8 inches which is ⅔ of a foot, so the amount of water needed to fill the kiddie pool to a depth of 8” is $6\sqrt{3} \times \frac{2}{3}$ or $4\sqrt{3}$ cubic feet.

20. This is another problem where there doesn’t seem to be enough information at first, but let’s jump in and try something anyhow. Suppose the side length of the smaller square is a and the side length of the larger square is b. Then the combined area of the two squares is $a^2 + b^2$.

How do we use the fact that VZ is 6 units in length? Well it is part of a right-angled triangle with side lengths a and b so we can use the Pythagorean Theorem:

$$a^2 + b^2 = 6^2$$

Look at that! We still don’t know the area or side length of either of the squares, but we can say with certainty that their areas must add up to 6^2 or 36 square units.

21. A little thought convinces us that 1 must go in the top left blank. How about the number 2? Well it could go either directly below or directly to the right of the number 1; we’ll consider these cases separately:

1 2 _ 1 _ _

_ _ _

There are two places the 3 can go, after which we can easily see how the remaining blanks can be filled (3 possible solutions):

123	124	125
456	356	346

2 _ _

In this case 3 must go directly to the right of 1, and there are two choices for 4 & 5, giving the following two solutions:

134	and	135
256		246

All told, we see that there are exactly 5 possible ways to legally arrange the numbers 1 through 6 in this puzzle.

22. We'll use the same algorithm as in the second solution for problem 3.7.2, counting paths to each junction on the network starting with those closest to A.

    Each intersection is reached from the either the intersection immediately above or immediately to its left, so the number of paths to an intersection is the sum of the number of paths to the intersection above and the number of paths to the intersection to its left.

    The diagram shows the network with these counts labeled, from which we can see that in our case there are exactly 26 such paths

23. First, notice that every rectangle contains exactly four right triangles. So let's count rectangles using this grid first.

    We'll count rectangles by size to make sure we don't miss any:

    - There are four 1×1 rectangles (so 4×4 = 16 associated right triangles)
    - There are two 1×2 rectangles and two 2×1 rectangles (giving another 16 right triangles)
    - There is one 2×2 rectangle, for another 4 right triangles.

    That accounts for all triangles whose legs are parallel to the rows and columns in this grid; are there any triangles at different angles?

Yes! Eight additional triangles have hypotenuse of length 2 along rows or columns of this grid with legs at a 45° angle: two examples are shown below (the other six are rotations of these:

Some thought convinces us that there are no right triangles offset at different angles within this grid.

Adding it all up, we see that there are exactly 44 right triangles which can be formed from vertices on this grid.

24. The mean is

$$\frac{1}{7}(7+77+777+7777+77777+777777+7777777)$$
$$=\frac{1}{7}\cdot 7+\frac{1}{7}\cdot 77+\frac{1}{7}\cdot 777+\frac{1}{7}\cdot 7777+\frac{1}{7}\cdot 77777+\frac{1}{7}\cdot 777777+\frac{1}{7}\cdot 7777777$$
$$=1+11+111+1111+11111+111111+1111111$$
$$=1234567$$

25. Since she averaged 12.5 points over the first 10 games, she must have scored a total of 10×12.5 = 125 points over these games.

    To average 15 points per game over the first 15 games, she needs to score a total of 15×15 = 225 points over these games.

    This means that she needs to score 100 points over the next five games, so she needs to average
    $\frac{100}{5}=20$ points per game over these games.

26. With a median of 4, the middle number must be a 4. With a unique mode of 5, there must be at least two 5's. So when arranged in numeric order we must have something like this:

    ? ? ? 4 5 5 ?

    Since 5 is a unique mode, there can't be any other repeated numbers. In particular, the three smallest numbers must be 1, 2, and 3 as they're all positive. So we know all but the last number:

1 2 3 4 5 5 ?

Finally, since the seven numbers average 4, they must add up to 4×7 = 28. The numbers so far add to 20, so the final number must be 8, which gives a range of 7 (i.e. 8 - 1).

27. Notice that the nth and nth-last terms of this sequence always add to $2x + 300$ for any number n, so each of these pairs averages $x + 150$. This means that the entire sequence has an arithmetic mean of $x + 150$, so $x + 150 = 180$ and therefore $x = 30$.