Math Team Notes
MSHSML Meet 4 Event C: Miscellaneous Topics

Summary
The purpose of these notes is to support mathlete preparation for participation in Minnesota State High School Mathematics League Meet 4, Individual Event C: Miscellaneous Topics. The notes primarily address the following newly introduced subtopics and are not comprehensive; therefore, mathletes are encouraged to review material beyond these notes, such as notes for prior meets and various textbooks. (And problems. Do lots and lots of problems.)

Subtopics
Topic 4C, Miscellaneous, includes the following subtopics.

4C	**Precalculus & Trigonometry: Miscellaneous Topics**
4C1	Sequences: patterns and recursion formulas, arithmetic and geometric sequences
4C2	Series: partial sums, formulas for sums of consecutive integers $1 + 2 + \cdots + n$, consecutive squares $1^2+2^2 + \cdots + n^2$, and consecutive cubes $1^3+2^3 + \cdots + n^3$.
4C3	Function notation
4C4	Factorial notation and the Binomial Theorem

Notes
Sequences
- **Definitions** A ***sequence*** is an ordered list of numbers. Each number in the sequence is a ***term***. A sequence can be ***finite*** $\{2, 4, 6, 8\}$ or ***infinite*** $\{5, 15, 45, 135, \ldots\}$.
- **Notation** Sequences can be written as functions, where the domain is the set of natural numbers $\{1, 2, 3 \ldots\}$ and the range is the set of terms. It is common to use the function name of a for a sequence with the term number, or domain, written as a subscript. Therefore a_1 is the first term of the sequence, a_2 is the second term, and a_n is the nth term. The sequence may be referred to as the sequence $\{a_1, a_2, a_3, \ldots\}$, or, more simply as $\{a_n\}$.
- **Notation** In some situations it is more convenient to use a domain of the set of whole numbers $\{0, 1, 2, 3 \ldots\}$, in which case a_0 would be the first term of the sequence, a_1 the second term, and so on. The equations in these notes assume the first term is a_1. The equations can be modified for when the first term is a_0.
- **Definitions** A ***recursive formula*** defines the nth term of a sequence in terms of the previous term; in other words, the a_n term is defined in terms of the a_{n-1} term. An ***explicit formula*** defines the nth term of a sequence in terms of n, meaning that the nth term can be found without knowing the previous term.
- **Definition** A ***periodic sequence*** is a sequence that repeats. The number of terms in the shortest part that repeats is the ***period***. For example, the sequence 2,4,8,2,4,8,2,4,8,… is a periodic sequence with period 3.

Arithmetic Sequences
- **Definitions** If the difference between consecutive terms in a sequence is a constant, then the sequence is an ***arithmetic sequence***. This constant is called the ***common difference***. It is found by starting with any term (except the first) and subtracting the previous term. The common difference is often represented by d and thus $d = a_n - a_{n-1}$, for $n \geq 2$. For example, the sequence $\{5, 8, 11, \ldots\}$ is an arithmetic sequence with a common difference of $d = 8 - 5 = 11 - 8 = 3$.
- **Theorem** The nth term of an arithmetic sequence is given by $a_n = a_1 + (n-1)d$, where d is the common difference. The right-hand side is simply the first term a_1 plus the number of common differences needed to get to the nth term. For example, the 20th term in the sequence $\{5, 8, 11, \ldots\}$ is $a_{20} = 5 + (20-1)2 = 5 + 38 = 43$. If the first term is a_0, the nth term of an arithmetic sequence is $a_n = a_0 + nd$.

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Geometric Sequences

- **Definitions** If the ratio between consecutive terms in a sequence is a constant other than 1, then the sequence is a ***geometric sequence***. This constant is called the ***common ratio.*** It is found by dividing any term (except the first) by its previous term. The common ratio is often represented by r and thus $r = \frac{a_n}{a_{n-1}}$, for $n \geq 2$. For example, the sequence $\{3, 12, 48, \ \ldots\}$ is a geometric sequence with a common ratio of $r = \frac{12}{3} = \frac{48}{12} = 4$.
- **Theorem** The nth term of a geometric sequence is given by $a_n = a_1 r^{n-1}$, where r is the common ratio. The right-hand side is simply the first term a_1 times the number of common ratios needed to get to the nth term. For example, the sixth term in the sequence $\{3, 12, 48, \ldots\}$ is $a_6 = 3(4^{6-1}) = 3(1024) = 3072$. If the first term is a_0, the nth term of a geometric sequence is $a_n = a_0 r^n$.

Series

- **Definitions** A ***series*** is the sum of the terms of a sequence. A series may be ***finite*** (if it has a finite number of terms) or ***infinite*** (if it has an infinite number of terms). The sum S of a finite sequence is the sum of its terms. An infinite series may or may not have a sum (some infinite series have a finite sum and some do not), but it does have partial sums. A ***partial sum*** is the sum of the first n terms, indicated by S_n. For example, the infinite arithmetic sequence $\{8, 5, 2, -1, \ldots\}$ does not have a sum, but $S_4 = 8 + 5 + 2 + (-1) = 14$.
- **Notation** A shorthand way of writing a series is to use ***summation notation***, sometimes called ***sigma notation***, because it uses the Greek upper-case letter sigma, Σ.
- **Notation** For the series $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$, the integer k is called the index, 1 is the lo***wer limit of summation***, and n is the ***upper limit of summation***.

Arithmetic Series

- **Definition** An ***arithmetic series*** is the sum of the terms of an arithmetic sequence.
- **Theorem** The sum of the first n terms of an arithmetic series is $S_n = \frac{n}{2}(a_1 + a_n)$.
- **Theorem** The sum of the first n natural numbers is $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. For example, the sum of the first ten natural numbers is $1 + 2 + \cdots + 10 = \frac{10(11)}{2} = 55$.
- [Insert TI-84 instructions for finding sums via the sum and seq functions.]

Geometric Series

- **Definition** A ***geometric series*** is the sum of the terms of a geometric sequence.
- **Theorem** The sum of the first n terms of a geometric series is $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$. Note that the formula gives a partial sum if the series is infinite. It gives the actual sum if the series is finite with n terms.
- **Definition** An infinite geometric series has infinitely many terms. The partial sums of some infinite geometric series can get closer and closer to a fixed number. The fixed number is called the ***limit*** and is considered the sum of the infinite series.
- **Definitions** An infinite geometric series only has a sum when the absolute value of the common ratio is between 0 and 1. These series are said to ***converge***. Conversely, an infinite geometric series does not have a sum when the absolute value of the ratio is greater than 1. These series are said to ***diverge***.
- **Theorem** The sum of an infinite geometric series is $S = \frac{a_1}{1-r}$, where r is the common ratio and $0 < |r| < 1$.

Sums of n, n^2, and n^3

- **Theorem** $\sum_{k=1}^{n} n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. This is an arithmetic series.
- **Theorem** $\sum_{k=1}^{n} n^2 = 1^2+2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. This is neither an arithmetic nor a geometric series.
- **Theorem** $\sum_{k=1}^{n} n^3 = 1^3+2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$. This is neither an arithmetic nor a geometric series.

Function Notation

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The Binomial Theorem

- **Definitions** The ***factorial*** of a positive integer is the product of all positive integers up to and including that integer. ***n factorial*** is denoted as $n!$ and is defined as $\boldsymbol{n}! = n(n-1)(n-2)\cdots 2\cdot 1$. Zero factorial is defined to be 1: $\mathbf{0}! = 1$. For example, 5 factorial is $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$.[1]
- **Definition** A ***combination*** is a selection of items where order does not matter. The ***combination of n objects taken r at a time*** is $\binom{\boldsymbol{n}}{\boldsymbol{r}} = \frac{n!}{r!(n-r)!}$. Alternate notations include $C(n, r)$ and $_nC_r$.
- **The Binomial Theorem** If n is a nonnegative integer, then

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$
$$= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r, \text{ where } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

For example, the expansion of $(2x+3)^5$ is

$$(2x+3)^5 = \binom{5}{0}(2x)^5(3)^0 + \binom{5}{1}(2x)^{5-1}(3)^1 + \binom{5}{2}(2x)^{5-2}(3)^2 + \binom{5}{3}(2x)^{5-3}(3)^3$$
$$+ \binom{5}{4}(2x)^{5-4}(3)^4 + \binom{5}{5}(2x)^{5-5}(3)^5$$
$$= (1)(32x^5)(1) + (5)(16x^4)(3) + (10)(8x^3)(9) + (10)(4x^2)(27) + (5)(2x)(81)$$
$$+ (1)(1)(243)$$
$$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$$

For example, the 8th term of $(5x+2)^{11}$ is

$$\binom{11}{8}(5x)^{11-8}(2)^8 = \left(\frac{11\cdot 10\cdot 9}{3\cdot 2\cdot 1}\right)(125x^3)(256) = 11\cdot 10\cdot 3\cdot 125x^3\cdot 128 = 5{,}280{,}000x^3$$

[where a calculator was used for the last step]
- **Tip** "Find the nth term using the Binomial Theorem" problems are common on exams (especially timed exams), as a correct answer demonstrates that you understand the theorem.

[1] This is one reason why in your writing you should be wary of writing something like, "When I asked the man how many children he had, he said he had 5!" (You probably don't mean to imply the man had 120 children.)

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- **Theorem** The coefficients of $(a+b)^n$ form a pattern known as Pascal's Triangle:

$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1
$(a+b)^5$	1 5 10 10 5 1
$(a+b)^6$	1 6 15 20 15 6 1
$(a+b)^7$	1 7 21 35 35 21 7 1
$(a+b)^8$	1 8 28 56 70 56 28 8 1
$(a+b)^9$	1 9 36 84 126 126 84 36 9 1
$(a+b)^{10}$	1 10 45 120 210 252 210 120 45 10 1

…

- **Note** Observe the following about Pascal's Triangle:
  - The 0th row is the single number 1.
  - The 1st row consists of the numbers 1 and 1.
  - Each subsequent row begins and ends with 1, and each middle number is the sum of the two numbers above it.
  - The first left and right diagonals are all 1's.
  - The second left and right diagonals are the natural numbers.
  - The third left and right diagonals are the triangular numbers.
- **Tip** You can use Pascal's Triangle as a quicker alternative to evaluating $\binom{n}{r}$ or as a check. For example, to evaluate $\binom{6}{4}$, write out the 6th row of Pascal's Triangle (the top row is the 0th row) as 1 6 15 20 15 6 1 and select the 4th number (the leftmost number is the 0th number), to obtain $\binom{6}{4} = 15$. Naturally, the better you know Pascal's Triangle, the better you will be able to do this. Of course, you can always create Pascal's Triangle from scratch, and, in time, you will know it well enough that you won't need to do this.

Problems

For the following problems, assume a calculator is not allowed unless stated.

Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. Determine exactly the value of this infinite geometric sum: $\frac{2}{125}+\frac{4}{625}+\frac{8}{3125}+\cdots$. [calculator allowed] {MSHSML 2019-20 4C #1)
2. Determine exactly the value of this infinite sum: $4+\frac{4}{3}+\frac{4}{9}+\cdots$. (MSHSML 2018-19 4C #1)
3. In the geometric sequence $\{a_n\}$, $a_2 = 8$ and $a_5 = 216$. Calculate the value of a_6. (MSHSML 2017-18 4C #1)
4. In arithmetic sequence $\{a_n\}$, $a_2 = 4$ and $a_{22} = 444$. Determine a_{222} exactly. [calculator allowed] (MSHSML 2016-17 4C #1)

Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. $a_1 = 3$, $a_2 = 6$, and $a_n = \frac{a_{n-1}}{a_{n-2}}$ is a periodic sequence with a period of 6. Determine exactly the value of a_{2020}. [calculator allowed] (MSHSML 2019-20 4C #2)
2. What is the value of the sum $1+2-3+4+5-6+7+8-9+\cdots+242-243$? (MSHSML 2018-19 4C #2)
3. What is the integer coefficient of the x^8 term in the expansion of $(2x^2-5)^7$? (MSHSML 2017-18 4C #2)
4. In geometric sequence $\{a_n\}$, $a_1 = 2$ and $a_3 \cdot a_4 = 5$.Determine exactly $a_5 \cdot a_6 \cdot a_7$. [calculator allowed] (MSHSML 2016-17 4C #2)

Problem #3 ("textbook with a twist"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. A function $f(x)$ is defined such that $f(-x) = -f(x)$ and $f(x) = f(x+6)$ for all x. If $f(1) = 5$, what is the value of $f(3) + f(5)$? [calculator allowed] (MSHSML 2019-20 4C #3)
2. If $f(x) = mx + b$, with $m < 0$, and $f(x) = 4f^{-1}(x) + 3$ for all x, compute the ordered pair (m, b). (MSHSML 2018-19 4C #3)
3. Calculate the integer value of this double sum: $\sum_{n=1}^{3}\sum_{b=1}^{10} b^n$. (MSHSML 2017-18 4C #3)
4. What is the smallest integer n for which $1^3+2^3+\cdots+n^3$ is a multiple of 77? [calculator allowed] (MSHSML 2016-17 4C #3)

Problem #4 ("challenge"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

1. If 5, 6, 9, and 15 are added, respectively, to four numbers constituting an increasing arithmetic progression, a geometric progression is formed. Determine exactly the four original numbers. [calculator allowed] (MSHSML 2019-20 4C #4)
2. When $\left(\sqrt{3}+\sqrt[3]{2}\right)^9$ is expanded, what is the sum of all the terms that are integers? (MSHSML 2018-19 4C #4)
3. For all positive integers n, $s(n)$ is defined as the sum of the digits of n. For example, $s(258) = 2+5+8 = 15$. What is the integer value of $s(1)+s(2)+s(3)+\cdots+s(1000)$? (MSHSML 2017-18 4C #4)
4. Determine the smallest $n > 2017$ for which $2\cdot3\cdot4+5\cdot6\cdot7+8\cdot9\cdot10+(3n-1)(3n)(3n+1)$ is a multiple of 27. [calculator allowed] (MSHSML 2016-17 4C #4)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge of other mathematical areas (algebra, for example). For more MSHSML Meet 4 Event C problems, see past exams, which date back to 1980-81.