Math Team Notes
Topic 4D: Analytic Geometry of the Conic Sections

Summary
The purpose of these notes is to support mathlete preparation for participation in Minnesota State High School Mathematics League Meet 4, Individual Event D: Algebra 2 & Analysis. The notes primarily address the following newly introduced subtopics, and are therefore not comprehensive; mathletes are encouraged to review material beyond these notes, such as notes for prior meets and various textbooks. (And problems. Do lots and lots of problems.)

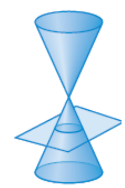
Subtopics
Topic 4D, Analytic Geometry of the Conic Sections, includes the following subtopics.

4D Algebra 2 & Analysis: Analytic Geometry of the Conic Sections
4D1 Use of the standard forms of equations of the conic sections
4D2 Graphs, including the location of foci, directrices, and asymptotes
4D3 Use of properties of conics to solve applied problems, including max-min problems using parabolas

Notes
All figures and tables in these notes are from *Algebra 2*, published by Saxon (2009).

Conic Sections

- **Definition** A ***conic section*** is a plane figure formed by the intersection of a double right cone and a plane. The four conic sections are the parabola, the circle, the ellipse, and the hyperbola. For example, in the figure, the intersection of the double right cone and a plane parallel to the base of a cone forms a circle (except when the plane intersects the shared apex of the cones).



Parabolas

- **Definition** A parabola is the set of all points equidistant from a point F, called the ***focus***, and a line l, called the ***directrix***, in a plane. The graph of a parabola is U-shaped, and the focus is on the "inside" of the graph and the directrix is on the "outside" of the graph. The ***axis of symmetry*** (or simply ***axis***) that divides the parabola into two congruent mirror images. The point at which the axis intersects the parabola is called the ***vertex*** of the parabola. Both the focus and the vertex lie on the axis, and the axis is perpendicular to the directrix.
- **Note** To visualize how an ellipse is a conic section, note that the intersection of the double right cone and a plane not parallel to the base of a cone and slanted sufficiently forms a parabola (except when the plane intersects the shared apex of the two cones).
- **Note** A parabola may open up, down, left, right, or any direction in general. The most commonly studied are those that open up or down, as these parabolas represent quadratic functions.
- **Definition** There are two standard forms of the equation of a parabola, depending upon whether the axis is horizontal or vertical. We start with the simpler case with the vertex at $(0,0)$. If the axis is horizontal, the ***standard form equation of a parabola with the center at*** $\mathbf{(0,0)}$ is $x = ay^2$; if the axis is vertical, the equation is $y = ax^2$.
- **Definition** We now consider the more general case when the vertex is (h, k) to present the two ***standard forms of the equation for a parabola***. If the axis is horizontal, the ***standard form equation of a parabola*** is $x = ay^2 + by + c$; if the axis is vertical, the equation is $y = ax^2 + bx + c$.

- **Theorem** Consider the special but common case of the parabola $y = ax^2$. The vertex is $(0,0)$. If $a > 0$ the parabola opens up and has a minimum value of 0; conversely, if $a < 0$ the parabola opens down and has a maximum value of 0. As $|a|$ decreases, the parabola becomes wider; conversely, as $|a|$ increases, the parabola becomes narrower. The equation for the axis of symmetry is $x = 0$, i.e., the y-axis. The focus is $\left(0, \frac{1}{4a}\right)$. The directrix has the equation $y = -\frac{1}{4a}$.
- **Theorem** Consider the special but common case of the parabola $y = ax^2 + bx + c$. If $a > 0$ the parabola opens up and has a minimum value; conversely, if $a < 0$ the parabola opens down and has a maximum value. As $|a|$ decreases, the parabola becomes wider; conversely, as $|a|$ increases, the parabola becomes narrower. The equation for the axis of symmetry is $x = -\frac{b}{2a}$.
- **Definition** There are two ***vertex forms of the equation of a parabola***, depending upon whether the axis is horizontal or vertical. If the axis is horizontal, the equation is $x - h = a(y-k)^2$; if the axis is vertical, the equation is $y - k = a(x-h)^2$. If the directrix is needed, then if the axis is horizontal, the equation is $x - h = \frac{1}{\pm 4p}(y-k)^2$, where $p = \frac{1}{4a}$; if the axis is vertical, the equation is $y - k = \frac{1}{\pm 4p}(x-h)^2$.
- [Define p and relate to directrix.]

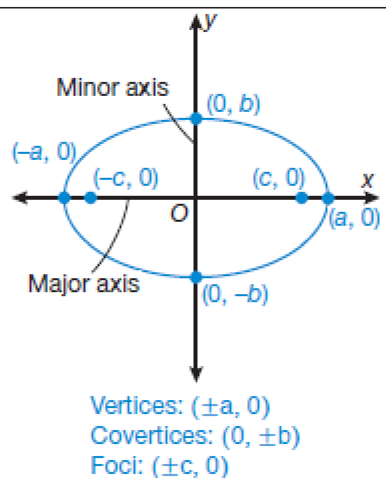
Circles

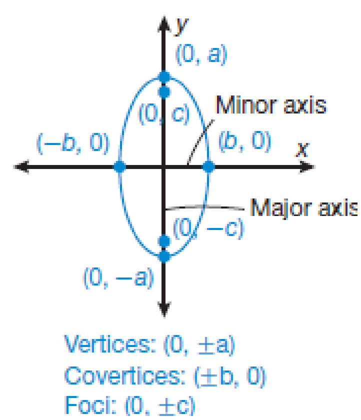
- **Definition** A ***circle*** is the set of all points in a plane that are at a given distance from a given point in the plane. A circle does not contain its center, and it does not include its interior. The center is often designated by the point O.
- **Note** To visualize how a circle is a conic section, note that the intersection of the double right cone and a plane parallel to the base of a cone forms a circle (except when the plane intersects the shared apex of the cones).
- **Definition** The ***standard form of the equation for a circle*** with center (h,k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$
- **Method** To graph a circle by hand on the coordinate plane,
  1. Identify and plot the center (h,k). (This point is not part of the circle.)
  2. Determine the radius r and plot a few points at a distance r from the center, such as the four points $(h \pm r, k)$ and $(h, k \pm r)$.
  3. Draw the circle by sketching between the points.
- **Review** Sometimes the center and the radius of a circle are not explicitly given, so it is necessary to use the Distance Formula and/or Midpoint Formula to determine these unknown parts. Given two points (x_1, y_1) and (x_2, y_2), the distance between them is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and their midpoint is $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- **Method** To determine the equation of a circle given the center and a point on the circle (x_1, y_1),
  1. Determine the radius r using the Distance Formula with (h,k) and (x_1, y_1).
  2. Substitute the center (h,k) and the radius r in the equation of a circle.
- **Method** To determine the equation of a circle given the endpoints of a diameter (x_1, y_1) and (x_2, y_2),
  1. Determine the center (h,k) using the Midpoint Formula with (x_1, y_1) and (x_2, y_2).
  2. Determine the radius r using the Distance Formula with either (h,k) and (x_1, y_1), or (h,k) and (x_2, y_2), or determine twice the radius $2r$ using the Distance Formula with (x_1, y_1) and (x_2, y_2) and divide by 2 to get r.
  3. Substitute the center (h,k) and the radius r in the equation of a circle.

<u>Ellipses</u>

- **Definitions** An ellipse is the set of all points P in a plane such that the sum of the distance from P to two fixed points F_1 and F_2 is constant. The two fixed points, F_1 and F_2, are called the ***foci*** (singular ***focus***). An ellipse has two axes. The ***major axis*** is the longer axis of the ellipse and passes through the foci. The endpoints of the major axis are the ***vertices*** of the ellipse. The ***minor axis*** is the shorter axis of the ellipse, and its endpoints are the ***co-vertices***. The major and minor axis are perpendicular, and their point of intersection (which is also the midpoint of the foci) is the ***center*** of the ellipse.
- **Note** To visualize how an ellipse is a conic section, note that the intersection of the double right cone and a plane not parallel to the base of a cone forms an ellipse (except when the plane intersects the shared apex of the two cones).
- **Definitions** There are two standard forms of the equation of an ellipse, depending upon whether the major axis is horizontal or vertical. We start with the simpler case with the center at $(0,0)$. If the major axis is horizontal, the ***standard form equation of an ellipse with the center at*** $(\mathbf{0},\mathbf{0})$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; if the major axis is vertical, the equation is $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$. See the table and figures.

Standard Form of the Equation of an Ellipse: Center at (0, 0)	
Major Axis Horizontal	Major Axis Vertical
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$
Vertices: $(a, 0), (-a, 0)$	Vertices: $(0, a), (0, -a)$
Foci: $(c, 0), (-c, 0)$	Foci: $(0\ c\ ), (0, -c)$
Co-vertices: $(0, b), (0, -b)$	Co-vertices: $(b, 0), (-b, 0)$





- **Definition** We now consider the more general case when the center is (h,k) to present the two ***standard forms of the equation for an ellipse***. If the major axis is horizontal, the equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$; if the major axis is vertical, the equation is $\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$. See the table.

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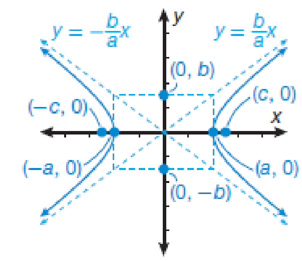
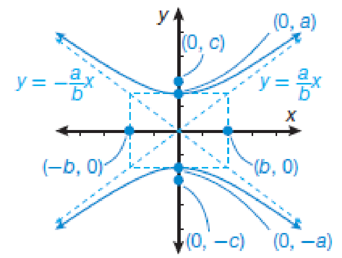
Standard Form of the Equation of an Ellipse: Center at (h, k)	
Major Axis Horizontal	Major Axis Vertical
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$
Vertices: $(h+a, k)$, $(h-a, k)$	Vertices: $(h, k+a)$, $(h, k-a)$
Foci: $(h+c, k)$, $(h-c, k)$	Foci: $(h, k+c)$, $(h, k-c)$
Co-vertices: $(h, k+b)$, $(h, k-b)$	Co-vertices: $(h+b, k)$, $(h-b, k)$

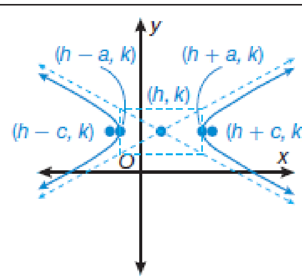
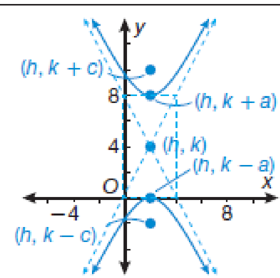
- **Corollary** The length of the major axis is $2a$, and the length of the minor axis is $2b$. The foci lie on the major axis.
- **Theorem** Given the equation of an ellipse as either $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ with foci $(h\pm c, k)$ or $\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$ with foci $(h, k\pm c)$ then, a, b, and c are related as $c^2=a^2-b^2$.
- **Theorem** Given the equation of an ellipse as either $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ or $\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$, the area of the ellipse is $ab\pi$.

Hyperbolas

- **Definition** A ***hyperbola*** (hy PER boh lah) is the set of all points P in a plane such that the difference of the distance from P to two fixed points F_1 and F_2, called the ***foci*** (singular ***focus***), is constant. The standard form of the equation of a hyperbola depends on the orientation of the hyperbola. The ***transverse axis*** is the segment that lies on the line containing the foci and has two endpoints, one on each ***branch*** of the hyperbola. The endpoints are the ***vertices*** of a hyperbola. The midpoint of the foci is the ***center*** of the hyperbola.
- **Note** To visualize how a hyperbola is a conic section, note that the intersection of the double right cone and a vertical plane forms a hyperbola (except when the plane includes the axis of the two cones).
- **Definitions** There are two standard forms of the equation of a hyperbola, depending upon whether the transverse axis is horizontal or vertical. We start with the simpler case with the center at $(0,0)$. If the major axis is horizontal, the ***standard equation of a hyperbola*** is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; if the major axis is vertical, the equation is $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$. See the table. Note that the a's correspond to the transverse axis (along with the c's), and the b's correspond do not.

Horizontal Orientation	Vertical Orientation
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
	
$c = \sqrt{a^2 + b^2}$, Eccentricity: $e = \sqrt{1 + \frac{b^2}{a^2}} = \frac{c}{a}$	
Foci: $(-c, 0)$ and $(c, 0)$ Vertices: $(-a, 0)$ and $(a, 0)$ Asymptotes: $y = \pm\frac{b}{a}x$	Foci: $(0, -c)$ and $(0, c)$ Vertices: $(0, a)$ and $(0, -a)$ Asymptotes: $y = \pm\frac{a}{b}x$

- **Definition** We now consider the more general case when the center is (h, k) to present the two ***standard forms of the equation for a hyperbola***. If the transverse axis is horizontal, the equation is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$; if the transverse axis is vertical, the equation is $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$. See the table.

Horizontal Orientation	Vertical Orientation
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
	

<u>Identifying Conic Sections</u>

- **Definitions** The ***standard forms for conic sections*** with center (h, k) are summarized in the following table.

Standard Forms for Conic Sections with Center (h, k)		
Circle	$(x-h)^2 + (y-k)^2 = r^2$	
	Horizontal Axis	**Vertical Axis**
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1;\ a > b$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1;\ a > b$
Hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Parabola	$x - h = \frac{1}{\pm 4p}(y-k)^2$	$y - k = \frac{1}{\pm 4p}(x-h)^2$

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- **Definition** The ***general form of the equation for a conic section*** is given by $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, and C are not all zero. (If A, B, and C were all zero, the equation would be linear.)
- **Definition** The quantity $B^2 - 4AC$ is called the ***discriminant*** of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. This is different from the $b^2 - 4ac$ discriminant used to discriminate quadratic equation solution types, but it has a nice familiarity in form and is also used to discriminate, but this time conic section types.
- **Summary** Methods for identifying conic sections in general form are given in the following table.

Identifying Conic Sections in General Form	
The general form of a conic section is given by $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, and C are not all zero.	
Conic Section	**Coefficients A, B, and C**
Circle	$B^2 - 4AC < 0$, $B = 0$, and $A = C$
Ellipse	$B^2 - 4AC < 0$ and either $B \neq 0$ or $A \neq C$
Hyperbola	$B^2 - 4AC > 0$
Parabola	$B^2 - 4AC = 0$

- **Tips**
  1. A parabola has only one squared term in its equation.
  2. A circle has both $B = 0$ and $A = C$.

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Problems

For the following problems, assume a calculator is not allowed unless stated.

Problem #1 ("quickie"; 1 point)

Goal: Know this topic so well that you can solve a Minnesota State High School Mathematics League (MSHSML) problem #1 in less than one minute.

1. A parabola has a minimum value of -7 and x-intercepts of -2 and 16. What are the coordinates of its vertex? (MSHSML 2019-20 4D #1)
2. What are the coordinates of the vertex of the parabola $y = 3x^2 - 12x + 7$? (MSHSML 2018-19 4D #1)
3. What are the coordinates of the focus of the parabola $y = (x+1)^2 - 1$? (MSHSML 2017-18 4D #1)
4. Determine exactly the vertex of $y = 2x^2 + 4x + 1$. (MSHSML 2016-17 4D #1)

Problem #2 ("textbook"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #2 in less than two minutes.

1. Determine exactly the distance between the vertices of the two parabolas determined by $y_1 = -x^2 + 2x$ and $y_2 = 2x^2 + 4x + 3$. (MSHSML 2019-20 4D #2)
2. A hyperbola has $y = \frac{5}{2}x + 24$ and $y = -\frac{5}{2}x + 4$ as its asymptotes and has a vertex at $(-4,19)$. What are the coordinates of the other vertex? (MSHSML 2018-19 4D #2)
3. What are the coordinates of the center of the conic $4x^2 - 9y^2 + 28x - 180y = 7$? (MSHSML 2017-18 4D #2)
4. An ellipse has center $(-2,7)$ and is tanget to both the x- and y-axes. If the minor axis is parallel to the x-axis, determine exactly the sum of the lengths of the major and minor axes. (MSHSML 2016-17 4D #2)

Problem #3 ("textbook with a twist"; 2 points)

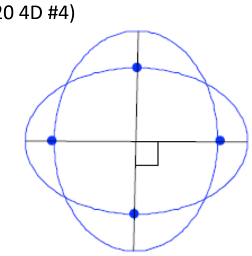
Goal: Know this topic so well that you can solve an MSHSML problem #3 in less than three minutes.

1. The ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in rectangle $ABCD$. A similar rectangle $PQRS$ is then inscribed in the ellipse. What is the positive difference of the areas of the two rectangles? (MSHSML 2019-20 4D #3)
2. For any real number m, the parabola $f_m(x) = 5x^2 + mx + 4m$ passes through a common point (a,b). Determine exactly the ordered pair (a,b). (MSHSML 2018-19 4D #3)
3. What is the greatest integer N for which the conic $x^2 + y^2 = 19x + 18y + N$ lies entirely in Quadrant I? (MSHSML 2017-18 4D #3)
4. Write an equation of the parabola with directrix $x = 4$ and focus at the origin. (MSHSML 2016-17 4D #3)

Problem #4 ("challenge"; 2 points)

Goal: Know this topic so well that you can solve an MSHSML problem #4 in less than six minutes.

1. Let $O = (0,0)$, $P = (-1,1)$, and Q be a point on the parabola $y = x^2 - 2$ in Quadrant I. If the area of $\triangle OPQ$ is 119, determine exactly the coordinates of Q. (MSHSML 2019-20 4D #4)
2. In the figure, two congruent ellipses have perpendicular major axes. Each ellipse passes through the other ellipse's foci. I the four foci are the vertices of a square with an area of 36, determine exactly the area of one of the ellipses. (MSHSML 2018-19 4D #4)



3. Determine exactly the value of k for which the conics $x^2 - 4y^2 = 1$ and $y^2 - kx^2 = 1$ intersect at points that all lie on a circle with an area of 111π. (MSHSML 2017-18 4D #4)

4. An isosceles triangle is constructed inside the ellipse $x^2 + 17y^2 - 4x - 68y + 4 = 0$. It has a vertex at each focus and the third vertex at a co-vertex (vertex on the minor axis) of the ellipse. Determine exactly the area of the isosceles triangle. (MSHSML 2016-17 4D #4)

If you are able to solve MSHSML problem #s 1, 2, and 3, in less than 1, 2, and 3 minutes, respectively, you will have at least 6 minutes (assuming a 12-minute, 4-question exam) to solve problem #4 ("challenge problem"; 2 points). Problem #4 tends to be more varied in nature than problems #1-3 and may require a broader knowledge of other mathematical areas (geometry, for example). For more MSHSML Meet 4 Event D problems, see past exams, which date back to 1980-81.