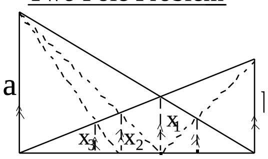
The History of "Noah Sheets"

The following four pages include a collection of interesting and useful mathematical formulas and relationships that were originally gathered together by IMSA alumnus, Noah Rosenberg, when he was a student at the Academy. He was an enthusiastic participant in math competitions and compiled them for use by our math team. His hand written notes have been edited and enhanced by Mr. George Milauskas (IMSA mathematics faculty). The resulting materials are affectionately known as the "Noah Sheets". Our hopes are that you will find some worthwhile ideas for use in math classes, mathlete training and problem solving. ✍

"The Noah Sheets" [Begun by Noah Rosenberg, IMSA Class of 1995, Edited and enhanced by George Milauskas]

Triangles

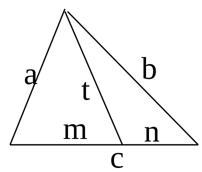
Two Pole Problem



$\frac{1}{x_1} = \frac{1}{a} + \frac{1}{b}$ so

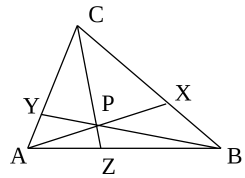
$x_1 = \frac{a \cdot b}{a + b}$ $x_k = \frac{a \cdot b}{a + kb}$

Stewart's Theorem



$a^2n + b^2m = t^2c + m \cdot n \cdot c$
(Proven by using Law of Cosines twice)

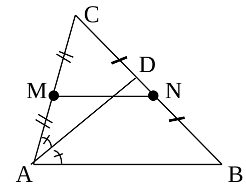
Ceva's Theorem



$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1$, $\frac{PX}{AX} + \frac{PY}{BY} + \frac{PZ}{CZ} = 1$

$\Leftrightarrow$ AX, BY, & CZ are concurrent

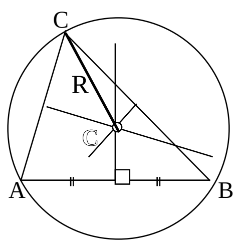
Basic Theorems



AC:AB = CD:DB (bis thm)

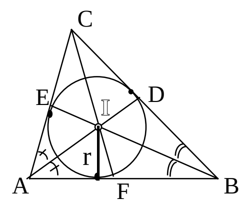
MN ∥ & $\frac{1}{2}$ AB [Midline Thm]

Circumcenter [− Bisectors]



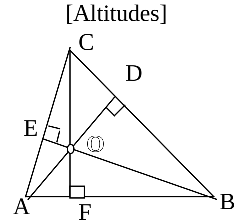
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Incenter [- Bisectors]



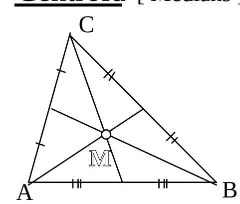
Area (ΔABC) $= \frac{1}{2}$ r P
$= r \cdot s$ (s=semi-perimeter)

Orthocenter [Altitudes]



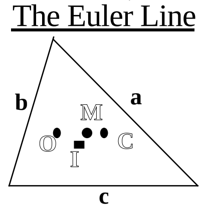
Watch for similar triangles.
eg: ΔADB ΔCFB

Centroid [Medians]



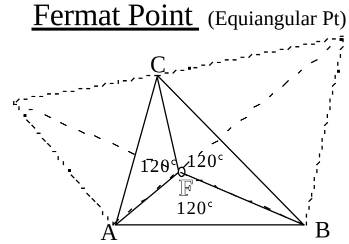
All six areas are equal.
$\mathbb{M}$ splits each median in ratio 2:1.
Coordinates of $\mathbb{M}$ = avg of vertices.

The Euler Line



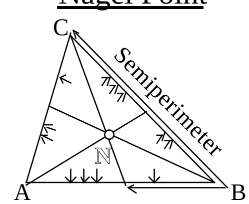
$\mathbb{O}$, $\mathbb{M}$, & $\mathbb{C}$ are collinear,
such that $\mathbb{OM} : \mathbb{MC} = 2:1$
and $9 \cdot (\mathbb{OC})^2 = a^2 + b^2 + c^2$

Fermat Point (Equiangular Pt)



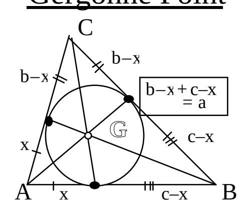
The sum A$\mathbb{F}$ + B$\mathbb{F}$ + C$\mathbb{F}$ is a minimum. (Found by, putting equilateral Δ's on sides)

Nagel Point



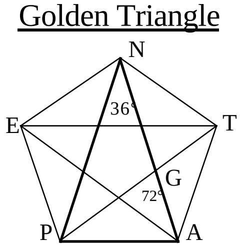
[Joins Semi-Perimeter Pts to Vertices]
Notice resulting segments

Gergonne Point



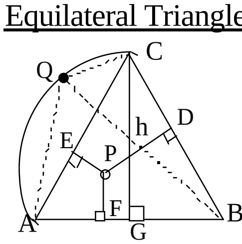
[Tangency Pts to Vertices]
Notice segments & "walkaround" labeling.

Golden Triangle



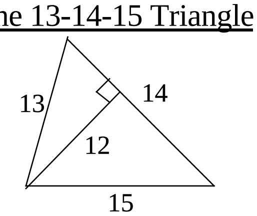
In a regular pentagon,
PN:PA = PG:GA = $\frac{1+\sqrt{5}}{2}$

Equilateral Triangles



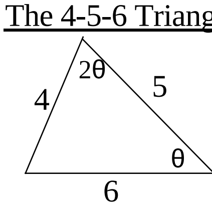
Sum of dist from any **P** to sides = **h**.
Any **Q** on Circum-⊙: QB = QC + QA
ΔCGB, has sides in a ratio, $1 : \sqrt{3} : 2$

The 13-14-15 Triangle



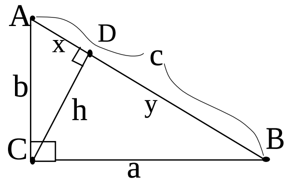
[An altitude and three sides are consecutive integers.]
Area = 84, r = 4, R = $\frac{65}{8}$

The 4-5-6 Triangle



One angle is twice the other.
Area = $6\sqrt{6}$

RIGHT TRIANGLES



$(a + b)^2 = c^2 + 4(\text{Area})$

$\frac{y}{a} = \frac{a}{c}$ $a^2 = y \cdot c$

$\frac{x}{b} = \frac{b}{c}$ $b^2 = x \cdot c$

$\frac{x}{h} = \frac{h}{y}$ $h^2 = x \cdot y$

$a \cdot b = c \cdot h$

$a^2 + b^2 = c^2$ Pythagorean Thm

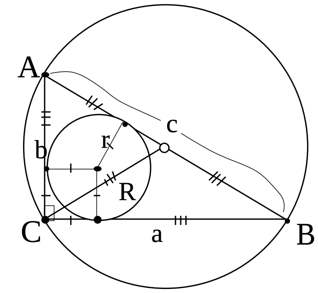
$\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2}$

$2 \cdot r = a + b - c$

$2 \cdot R = c$ $\frac{a+b}{2} = R + r$

median = $\frac{1}{2}$ hypotenuse

$(m^2 - n^2, 2mn, m^2 + n^2)$ is Pythagorean triple for m, n positive integers



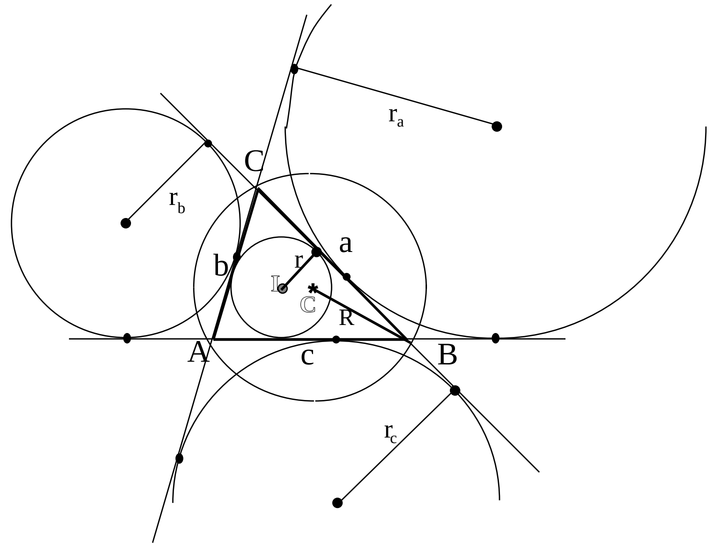
A Triangle And Its Circles: PROPERTIES

Δ ABC has sides: **c**, **b**, and **a**,
and angles A, B, and C.

The radii of the:
- Inscribed circle, **r**.
- The three escribed circles: $\mathbf{r_a}$, $\mathbf{r_b}$, & $\mathbf{r_c}$
- and the circumscribed circle, **R**.

The area of the triangle is **K**.

The semiperimeter is **S**.



Drawn to sides **a**, **b**, and **c**, respectively:
- Let $\mathbf{m_a}$, $\mathbf{m_b}$, & $\mathbf{m_c}$ be the medians.
- Let $\mathbf{t_a}$, $\mathbf{t_b}$, & $\mathbf{t_c}$ be the angle bisectors.
- Let $\mathbf{h_a}$, $\mathbf{h_b}$, & $\mathbf{h_c}$ be the altitudes.

The following relationships are true for triangles as labeled above:

$K = \sqrt{S\,(S-a)\,(S-b)\,(S-c)}$ Heron's Formula

$$K = \frac{1}{2}\cdot a\cdot b\cdot \sin C = \frac{a^2 \sin B \cdot \sin C}{2 \sin A} = r\cdot S = \frac{a\cdot b\cdot c}{4\cdot R}$$

$$R = \frac{a\cdot b\cdot c}{4\cdot K} \qquad 2\cdot R\cdot r = \frac{a\cdot b\cdot c}{a+b+c} \qquad 2\cdot r \le R \text{ in all } \Delta\text{'s}$$

$$t_c = \frac{2\cdot a\cdot b\cdot \cos\frac{C}{2}}{a+b} = \frac{2\sqrt{a\cdot b\cdot S\cdot (S-c)}}{a+b}$$

$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \qquad r^2 = \frac{(s-a)(s-b)(s-c)}{s}$$

$c^2 = 2a^2 + 2b^2 - 4m_c^2$ and its permutations

Law of Cosines:
$c^2 = a^2 + b^2 - 2\cdot a\cdot b \cos C$ (& permutations)

Law of Sines:
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Law of Tangents:
$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$$

Other

Point-Line(plane) Distance

between (x_0,y_0) and line $ax + by + c = 0$:

$$\text{dist is: } \frac{|\,a\cdot x_0 + b\cdot y_0 + c\,|}{\sqrt{a^2 + b^2}}$$

between (x_0,y_0,z_0) and line $ax + by + c\cdot z + d = 0$:

$$\text{is: } \frac{|\,a\cdot x_0 + b\cdot y_0 + c\cdot z_0 + d\,|}{\sqrt{a^2 + b^2 + c^2}}$$

also dist = $\dfrac{a\cdot b\cdot c}{a\cdot b+b\cdot c+a\cdot c}$

Logarithms

$$\log_b N = p \quad \Leftrightarrow \quad b^p = N$$

$$\log_b N = \frac{\log_a N}{\log_a b} \quad \text{"change base"}$$

$$\log \frac{m\cdot n}{q} = \log m + \log n - \log q$$

$$\log N^p = p \log N$$

$$\log_b a = \frac{1}{\log_a b}$$

Series

Series: $S = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

•Arithmetic: Constant Difference $d = a_{n+1} - a_n$

$$a_n = a_1 + (n-1)\cdot d \quad \& \quad S_n = \frac{n}{2}(a_1 + a_n)$$

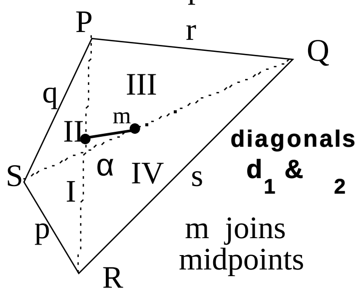
•Geometric: Constant ratio $r = \dfrac{a_{n+1}}{a_n}$

$$a_n = a_1\cdot r^{n-1}, \quad S_n = \frac{a - a\,r^{n-1}}{1-r} \quad \& \quad S_\infty = \frac{a_1}{1-r}$$

$(-1 < r < 1)$

If you find any errors, or have any worthy additions to "The Noah Sheets" please contact George Milauskas, Mathematics Coordinator at the Illinois Mathematics and Science Academy, 1500 Sullivan Rd, Aurora, Illinois, 60506 (708)907-5965: E-mail: geom@imsa.edu

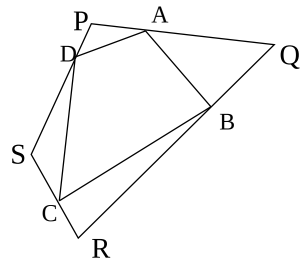
Quadrilateral Properties: **K** = Area, **r** = inradius, **R** = circumradius, **P** = perimeter, **S** = semiperimeter



Areas: $A_I \cdot A_{III} = A_{II} \cdot A_{IV}$

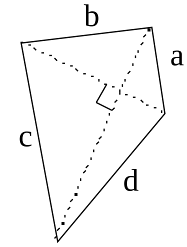
$\mathbf{K}_{PQRS} = \frac{1}{2} d_1 d_2 \sin \alpha$

$p^2 + q^2 + r^2 + s^2 = d_1^2 + d_2^2 + (2m)^2$



A,B,C,D are midpoints, ABCD is a parallelogram.

If $\frac{PA}{AQ} = \frac{QB}{BR} = \frac{RC}{CS} = \frac{SD}{DP} = n$ then the ratio of areas, $\mathbf{K}(ABCD) : \mathbf{K}(PQRS) = \frac{n^2+1}{(n+1)^2}$



If Diagonals are perpendicular,

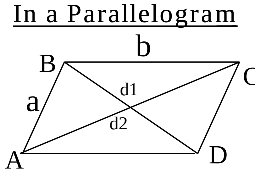
$\mathbf{K} = \frac{1}{2}(\text{diag }1)(\text{diag }2)$

$a^2 + c^2 = b^2 + d^2$

⇔ (if and only if)

the diagonals are .

In a Parallelogram



$2(a^2 + b^2) = d_1^2 + d_2^2$

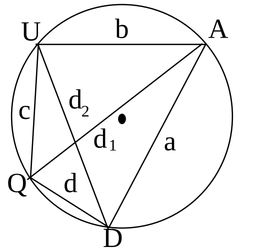
$BD = a^2 + b^2 - 2ab \cos A$

K = a · b · sin A .

$\frac{d_1}{d_2} = \frac{ab + cd}{ad + bc}$

(2nd Ptolemy's Theorem)

Cyclic Quadrilaterals

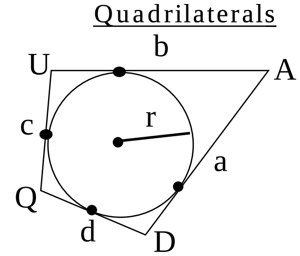


A + Q = U + D = 180°

$d_1 \cdot d_2 = a \cdot c + b \cdot d$ (Ptolemy)

$\mathbf{K} = \sqrt{(S-a)(S-b)(S-c)(S-d)}$

Circumscribed Quadrilaterals

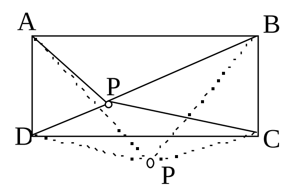


a+c = b+d ⇔ QUAD has in⊙

$r = \frac{\mathbf{K}}{a+c}$ and $\mathbf{K} = \frac{1}{2}\, \mathbf{r} \cdot \mathbf{P}$

If QUAD is both inscribed. and circumscr., then $K = \sqrt{a \cdot b \cdot c \cdot d}$

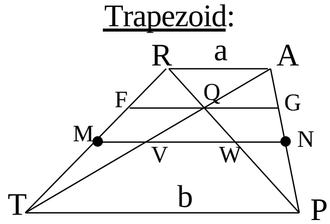
The Rectangle



For any point, P, & rectangle ABCD

$(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$

Trapezoid:



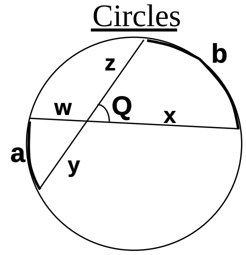
If MN = median, $MN = \frac{a+b}{2}$

$VW = \frac{b-a}{2}$ $\quad FG = \frac{2 \cdot a \cdot b}{a+b}$

If $\overline{MN}$ is ***any*** parallel,

$MN = \frac{RM \cdot b + MT \cdot a}{RT}$

Circles

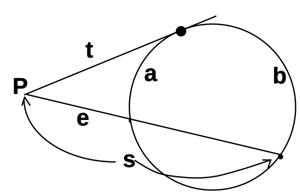


Angle-Arc Property

$Q = \frac{b + a}{2}$

Power Theorem

$w \cdot x = y \cdot z$

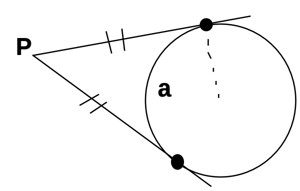


Angle-Arc Property

$P = \frac{b - a}{2}$

Power Theorem

$t^2 = s \cdot e$



A tangent to a circle is perpendicular to a radius.

Two tangents to a circle from an outside point are equal.

tan-tan angle = supp of inner arc. P + a = 180°

Volume and Surface Areas of Solids:

Prismatic solids: (prism, box, cylinder)

Lateral Area = (base perimeter)(height)

Total Area = lateral area + 2 bases

Volume = (Area of Base)(height)

Pointed Solids: (pyramid, cone)

[Linearly related cross sections]

Lateral Area (add lat faces), $LA_{(cone)} = \pi \cdot r \cdot l$

l = lateral edge (slant height)

Total Area = lateral area + one base

Volume = $\frac{1}{3}$(Area of Base)(height)

Spheres:

Total Surface Area = $4 \cdot \pi \cdot r^2$

Volume = $\frac{4}{3} \pi r^3$

Ellipsoid: Volume = $\frac{4}{3}\, a \cdot b \cdot c \cdot \pi$

Prismoidal Volume Formula: $V = \frac{h}{6}(B_1 + 4M + B_2)$

[For solids with quadratically related cross sections, height h, upper bases B_1, B_2 and mid section M]

Trigonometry:

$\sin A = \frac{\text{opp leg}}{\text{hypotenuse}}$

$\cos A = \frac{\text{adj leg}}{\text{hypotenuse}}$

$\tan A = \frac{\text{opp leg}}{\text{adj leg}} = \frac{\sin A}{\cos A}$

$\csc A = \frac{\text{hypotenuse}}{\text{opp leg}} = \frac{1}{\sin A}$

$\sec A = \frac{\text{hypotenuse}}{\text{adj leg}} = \frac{1}{\cos A}$

$\cot A = \frac{\text{adj leg}}{\text{opp leg}} = \frac{1}{\tan A}$

Values to Memorize:

$\sin 30° = \frac{1}{2} = \cos 60°$

$\cos 30° = \frac{\sqrt{3}}{2} = \sin 60°$

$\tan 30° = \frac{\sqrt{3}}{3} = \cot 60°$

$\sin 45° = \cos 45° = \frac{\sqrt{2}}{2}$

$\tan 45° = 1$

$\sin 15° = \frac{\sqrt{6} - \sqrt{2}}{4} = \cos 75°$

$\cos 15° = \frac{\sqrt{6} + \sqrt{2}}{4} = \sin 75°$

$\tan 15° = 2 - \sqrt{3}$, $\tan 75° = 2 + \sqrt{3}$

Golden rectangle & regular pentagon.

$\sin 18° = \cos 72° = \frac{\sqrt{5} - 1}{4}$

$\cos 36° = \sin 54° = \frac{\sqrt{5} + 1}{4}$

Pythagorean Identities

$\sin^2 A + \cos^2 A = 1$

$1 + \tan^2 A = \sec^2 A$

$1 + \cot^2 A = \csc^2 A$

Odd-Even Functions:

$\sin(-A) = -\sin(A)$

$\cos(-A) = \cos(A)$

$\tan(-A) = -\tan(A)$

Complements A & B

$\sin^2 A + \sin^2 B = 1$

$\sin A = \cos B$, etc

Sum to Product

$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}$

$\sin A - \sin B = 2 \sin \frac{A - B}{2} \cos \frac{A + B}{2}$

$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}$

$\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}$

$\tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A \cdot \cos B}$

Product to Sum

$\sin A \cdot \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$

$\cos A \cdot \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$

$\sin A \cdot \cos B = \frac{1}{2} [\sin(A-B) + \sin(A+B)]$

$\tan A \cdot \tan B = \frac{\cos(A-B) - \cos(A+B)}{\cos(A-B) + \cos(A+B)}$

Sum & Difference Identities

$\sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \sin B$

$\cos(A \pm B) = \cos A \cdot \cos B \mp \sin A \cdot \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Double Angle Identities

$\sin 2A = 2 \sin A \cdot \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

or $= 1 - 2\sin^2 A = 2\cos^2 A - 1$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

Triple Angle Identities

$\sin 3A = 3 \sin A - 4 \sin^3 A$

$\cos 3A = 4 \cos^3 A - 3 \cos A$

$\tan 3A = \frac{\tan A \cdot (\tan^2 A - 3)}{3 \tan^2 A - 1}$

Half Angle Formulas

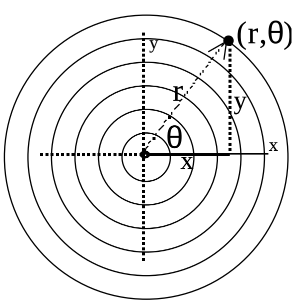
$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}$

$\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}$

$\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A} = \frac{1 - \cos A}{\sin A}$

Polar Coordinates

Points are represented in terms of (r,θ) rather than (x,y)



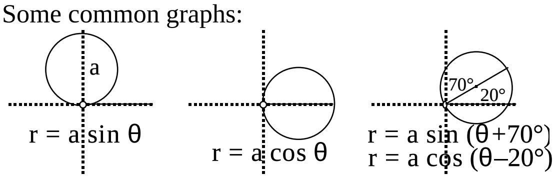
$x^2 + y^2 = r^2$

$\tan \theta = \frac{y}{x}$

$x = r \cos \theta$

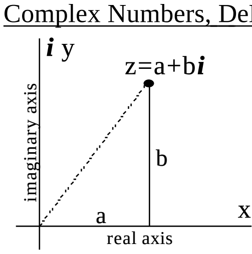
$y = r \sin \theta$

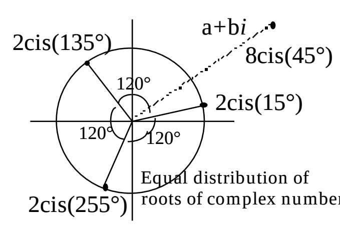
Some common graphs:



$r = a \sin \theta$ $r = a \cos \theta$ $r = a \sin(\theta + 70°)$ $r = a \cos(\theta - 20°)$

Complex Numbers, DeMoivre's Thm, Euler's Thm & C*I* S





$Z = a + b\boldsymbol{i} = r \operatorname{cis} \theta$ (polar form of complex number)

The magnitude, $r = |a + b\boldsymbol{i}| = \sqrt{a^2 + b^2}$

$e^{\boldsymbol{i}\theta} = \cos\theta + \boldsymbol{i} \sin\theta = \operatorname{cis}\theta$ (Euler)

$\operatorname{cis}(A + B) = \operatorname{cis} A \cdot \operatorname{cis} B$ $\operatorname{cis}(A - B) = \frac{\operatorname{cis} A}{\operatorname{cis} B}$

DeMoivre's Theorems: (see illustration above)

$(a + b\boldsymbol{i})^n = (r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n \cdot \theta)$ for n = pos int

$\sqrt[n]{r \cdot \operatorname{cis} \theta} = \sqrt[n]{r} \cdot \operatorname{cis}\left(\frac{2\pi k + \theta}{n}\right)$ for k = 0,1,2,3...,n–1