Chapter 2

Complex Numbers

2.1 The Square Root of −1

The study of complex numbers begins when we are bold enough to ask a very childish question: what is the square root of −1? Forbidden by sixth grade teachers the world over, the expression $\sqrt{-1}$ is nevertheless the key to a whole branch of math.

Historically, people were led to write $\sqrt{-1}$ by the quest to solve equations. Clearly, an equation like

$$x^2 + 1 = 0 \tag{2.1}$$

has only the solutions $x = \pm\sqrt{-1}$. In order to be able to solve *all* equations, it was decided to accept $\sqrt{-1}$ as a legitimate number.

The square root of −1 is usually written as i. This weird number shows its weird properties almost from the beginning, as we shall see. It is not a real number in the mathematical sense. This is not to say it is not real, at least any less than negatives are. If we multiply i by a real number like 2 or π, we get a number like $2i$ or πi; there is no way to simplify this product. Numbers like this, formed by multiplying i by a real, are called **pure imaginary numbers**, though you should not let this prejudice of name keep you from accepting them as regular numbers. Treat the word imaginary as a purely mathematical definition.

If we multiply i by itself, we get $\sqrt{-1}\sqrt{-1} = (\sqrt{-1})^2 = -1$, as we would expect. But notice that if we try to combine the radicals and write $\sqrt{-1}\sqrt{-1} = \sqrt{(-1)^2} = 1$, we will get the wrong answer. Manipulations like this are forbidden.

If we keep taking powers of i we get $i^3 = ii^2 = -i$, $i^4 = ii^3 = i(-i) = -i^2 = 1$, $i^5 = ii^4 = i$, $i^6 = -1$, etc. The powers of i go in cycles of 4: $i, -1, -i, 1, i, -1, -i, 1$, etc.

EXERCISE 2-1 What is i^{17}? How about i^{69}? i^{1972}?

2.2 Complex Number Operations

The so-called **complex numbers** are just the numbers you get when you add a real to an imaginary, like $\sqrt{2} + 3i$ or $-17 + \frac{17}{2}i$. Every real number is also a complex number; the imaginary component is just 0. Those complex numbers which are not real are called **imaginary numbers**. (This is not exactly the same as pure imaginary numbers; can you write a number which is imaginary but not pure imaginary?)

EXAMPLE 2-1 Let's clear up these confusing definitions by looking at some examples. 3 is both real and complex, but not imaginary. $3i$ is not real, but is complex, imaginary, and pure imaginary. $3+3i$ is neither real nor pure imaginary, but is imaginary and complex. (We realize this is unnecessarily complicated, but they *are* called complex numbers. . .)

Complex variables are usually designated by z or w, for no other reason than that letters near the end of the alphabet are best for variables, and x and y are already typically used for reals.

To add two complex numbers together, all we have to do is add their real and imaginary parts separately, as in the following examples.

EXAMPLE 2-2 Let's add $3+4i$ to $-3+8i$. The sum is just $3-3+4i+8i = \mathbf{12i}$.

EXERCISE 2-2 What is $\left(-\frac{1}{4}+i\right)+\left(2-\frac{3}{4}i\right)$?

EXERCISE 2-3 Find the general formula for the sum $(z_1+z_2i)+(w_1+w_2i)$.

Subtraction follows easily from addition. Furthermore, we can multiply two complex numbers with the distributive law.

EXAMPLE 2-3 Let's multiply $3+4i$ by $-3+8i$. The product is

$$\begin{aligned}(3+4i)(-3+8i) &= 3(-3+8i)+4i(-3+8i)\\ &= (3)(-3)+(3)(8i)+(4i)(-3)+(4i)(8i)\\ &= -9+24i-12i-32 = \mathbf{-41+12i}.\end{aligned}$$

(Note the negative sign of the 32; it comes from i times i.)

EXERCISE 2-4 What is $\left(-\frac{1}{4}+i\right)\left(2-\frac{3}{4}i\right)$?

EXERCISE 2-5 Find the general formula for the product $(z_1+z_2i)(w_1+w_2i)$.

EXERCISE 2-6 Simplify $(z_1+z_2i)(z_1-z_2i)$.

When we divide two complex numbers, we clear all instances of i from the denominator in exactly the same way as rationalizing a denominator which contains square roots. We use the fact that the complex number $a+bi$ multiplied by $a-bi$ is real, just as $a+\sqrt{b}$ multiplied by $a-\sqrt{b}$ gets rid of the square root. (You showed this in Exercise 2-6 above, right?)

EXAMPLE 2-4 Let's divide $3+4i$ by $-3+8i$. The quotient is

$$\frac{3+4i}{-3+8i} = \frac{3+4i}{-3+8i}\cdot\frac{-3-8i}{-3-8i} = \frac{23-36i}{73} = \frac{23}{73} - \frac{36}{73}i.$$

EXERCISE 2-7 What is $\dfrac{-\frac{1}{4}+i}{2-\frac{3}{4}i}$?

EXERCISE 2-8 Find the general formula for the quotient $(z_1+z_2i)/(w_1+w_2i)$.

We can do more complicated operations, like taking square or cube roots of complex numbers, but we'll let that wait for now. We should define a couple of basic notations, however. Consider an arbitrary complex number $z = a+bi$. We denote the number $a-bi$ by $\overline{z}$, and call it the **conjugate** of z. We call the number a the **real part** of z, and denote it by $\mathrm{Re}(z)$. Similarly, the number b is the **imaginary part** of $a+bi$. WARNING: The imaginary part of $a+bi$, denoted $\mathrm{Im}(a+bi)$, refers to the real number b, not to bi. Thus $\mathrm{Im}(a+bi) = b$, NOT bi.

EXERCISE 2-9 Prove that $\overline{\overline{z}} = z$ for all complex z.

EXERCISE 2-10 What is the conjugate of a real number a? of a pure imaginary number bi?

EXERCISE 2-11 Show that $\overline{z+w} = \overline{z}+\overline{w}$ for all z and w. Does this fact surprise you?

EXERCISE 2-12 Show that $\overline{zw} = \overline{z}\,\overline{w}$ for all z and w. Does this surprise you?

EXERCISE 2-13 How about $\overline{(z/w)}$? Surprising?

EXAMPLE 2-5 Consider $\mathrm{Im}(z)+\mathrm{Im}(\overline{z})$. Let $z = a+bi$, so that $\overline{z} = a-bi$. Then $\mathrm{Im}(z) = b$ and $\mathrm{Im}(\overline{z}) = -b$, so that $\mathrm{Im}(z)+\mathrm{Im}(\overline{z}) = 0$, no matter what z is.

EXERCISE 2-14 What is $\mathrm{Re}(z) + i\,\mathrm{Im}(z)$?

Problems to Solve for Chapter 2

17. Find $\dfrac{1+i}{3-i}$. (MAΘ 1987)

18. Which are true? (MAΘ 1987) (Don't look back at the text!)

i) $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$
ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
iii) $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$

19. Evaluate $\sqrt{-1}\left(\sqrt{-1}\right)^2\sqrt{(-1)^2}$. (MAΘ 1991)

20. Find $i^{-18}+i^{-9}+i^{0}+i^{9}+i^{18}$. (MAΘ 1991)

21. If a, b, c, and d are real, then find $\text{Re}\,[(a + bi)(c + di)]$ in terms of a, b, c, and d. (MAΘ 1991)

22. Evaluate $(2 + i)^3$. (MAΘ 1991)

23. Find $(1 + i)^4(2 - 2i)^3$. (MAΘ 1987)

24. Simplify $\dfrac{\sqrt{-6}\,\sqrt{2}}{\sqrt{3}}$. (MAΘ 1990)

25. If $F(x) = 3x^3 - 2x^2 + x - 3$, find $F(1 + i)$. (MAΘ 1990)

26. Which of the following are true? (MAΘ 1987)

i) $\overline{z + 3i} = z - 3i$
ii) $\overline{iz} = -i\overline{z}$
iii) $(2 + i)^2 = \overline{3 - 4i}$