Chapter 13

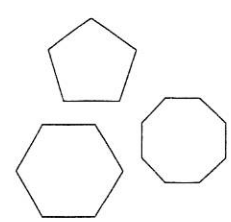
Polygons

13.1 Types of Polygons

A **polygon** is a simple closed planar figure formed by line segments. Polygons are classified by the number of sides they have; we have already discussed triangles, which have three sides, and quadrilaterals, which have four sides. A polygon with n sides is generically called an n-gon, but many types of polygons have special names as well. The most common ones are shown in the table below.

# sides	Name
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon
12	dodecagon

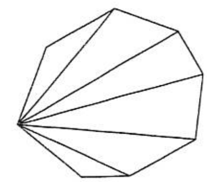
A polygon is called a **regular polygon** if all of its sides are equal and all of its angles are equal. Remember, as we saw with quadrilaterals, just because all the sides of a polygon are equal doesn't mean the polygon is regular. The same is true of the angles of a polygon. (Can you draw a polygon whose interior angles are equal but which is still not regular?)



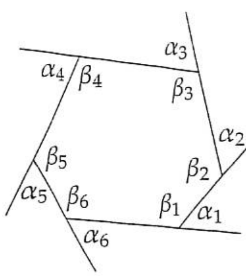
As with quadrilaterals, any segment drawn from one vertex to a non-adjacent vertex is called a diagonal.

To count the number of diagonals in an n-gon, we can count the number of diagonals from each vertex. Each vertex can be connected to $n-1$ other vertices. Two of these segments form sides, while the other $n-3$ form diagonals. Since there are n vertices, there are a total of $n(n-3)$ diagonals. To test this formula, consider a quadrilateral, for which $n = 4$. Our formula gives $4(4-3) = 4$ diagonals, not 2! Obviously we have overlooked something. In our counting method we have actually counted every diagonal twice, once for each endpoint. Thus, to get an accurate count of the diagonals, we must divide by 2, leaving $n(n-3)/2$ diagonals in an n-gon.

13.2 Angles in a Polygon



To determine the sum of the angles in a polygon, we divide the polygon into triangles just as we did for the quadrilateral. Draw the $n-3$ diagonals from one vertex. This divides the polygon into $n-2$ triangles as shown. Adding up all the angles in these triangles gives the sum of the angles of a polygon with n sides: $180(n-2)$ degrees.



Now, if we consider the exterior angles of a polygon as shown at left, we see that if we let the interior angle at vertex i equal β_i and the exterior angle be α_i, then at each vertex we have $\alpha_i + \beta_i = 180°$. If we add these equations together for all n vertices and group the interior and exterior angles together, we get

$$\left(\alpha_1+\beta_1\right)+\cdots+\left(\alpha_n+\beta_n\right)=(\alpha_1+\cdots+\alpha_n)+(\beta_1+\cdots+\beta_n)=180n.$$

Using what we know about the sum of the interior angles, we have

$$(\alpha_1+\cdots+\alpha_n)+(\beta_1+\cdots+\beta_n)=(\alpha_1+\cdots+\alpha_n)+180(n-2)=180n.$$

Thus,

$$(\alpha_1+\cdots+\alpha_n)=180n-180(n-2)=360,$$

and the sum of the exterior angles of any polygon is 360°.

13.3 Regular Polygons

Most polygons you will encounter in geometry problems which have more than four sides will be regular polygons. As we stated before, the angles of a regular polygon are all equal, so knowing the sum of the interior angles and the sum of the exterior angles, we can determine the measure of each angle in a regular polygon.

$$\text{Interior angle} = \frac{180(n-2)}{n} = 180 - \frac{360}{n}$$

$$\text{Exterior angle} = \frac{360}{n}$$

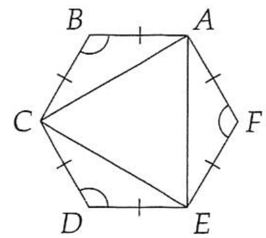
Below is a table of the interior angle measures of some common regular polygons.

# Sides	Angle	# Sides	Angle
3	60°	8	135°
4	90°	9	140°
5	108°	10	144°
6	120°	12	150°

It's not necessary to memorize these; just be familiar with them.

EXAMPLE 13-1 Prove that by connecting every other vertex of a regular hexagon, we form an equilateral triangle.

Proof: To show this, we must merely show that the three sides AC, CE, and AE, are equal in the figure below.



By SAS we have

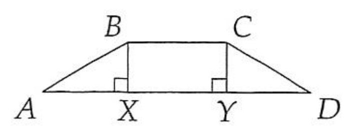
$$\triangle ABC \cong \triangle CDE \cong \triangle EFA.$$

The sides in question are corresponding sides of these triangles, so we have $AC = CE = AE$ and thus $\triangle ACE$ is equilateral.

EXAMPLE 13-2 Find the number of sides in a polygon whose interior angles have sum 2340°.

Solution: A polygon with n sides has interior angle sum $180(n-2)$. Solving $180(n-2) = 2340$, we find that $n = \mathbf{15}$.

EXAMPLE 13-3 Given that $ABCD \cdots L$ is a regular dodecagon, find the length of AD if $AB = 4$.

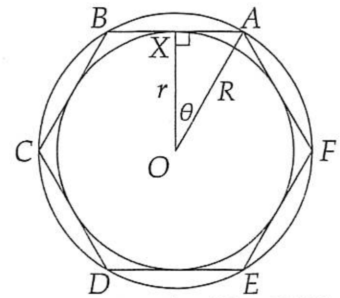


Solution: We attack this problem as we do most geometry problems: cut the problem into quadrilaterals and triangles. The diagram only shows the relevant portion of the dodecagon. Since a dodecagon has 12 sides, each interior angle has measure 150°. Since $ABCD$ is an isosceles trapezoid (why?), we have $\angle BAX = \angle CDY = 180° - \angle BCD = 30°$. Drawing BX and CY perpendicular to AD, we find that $BCYX$ is a rectangle because $\angle YCD = 90° - \angle CDY = 60°$, so $\angle BCY = 150° - 60° = 90°$. Thus, $XY \doteq BC = 4$. Since $\angle BAX = 30°$ in right triangle ABX, we have $BX = AB/2 = 2$ and $AX = BX\sqrt{3} = 2\sqrt{3}$. Similarly $YD = 2\sqrt{3}$. Thus $AD = AX + XY + YD = \mathbf{4 + 4\sqrt{3}}$.

EXERCISE 13-1 Find the number of sides in a regular polygon which has interior angle measure 162°.

EXERCISE 13-2 Prove that we form a square if we connect every other vertex of a regular octagon. (Remember: just showing the sides are equal does not mean the quadrilateral is a square.)

Just as with triangles, the perpendicular bisectors of the sides of a regular polygon all pass through a single point. (Can you use triangle congruence to prove this fact?) Furthermore, the angle bisectors of the interior angles also meet at this point. Thus, we can construct both a circle which passes through the vertices of the polygon and a circle which is tangent to all the sides of the polygon. We have drawn these circles below with radii R and r, respectively.



As shown in the figure, these two circles are concentric. If the polygon has side length l and n sides, then we can determine the inradius (sometimes called the **apothem**), r, the circumradius, R, and the area of the polygon by breaking the polygon into right triangles like $\triangle AXO$ at right. Remember that the radius of the circle is perpendicular to a tangent at the point of tangency, so $\angle OXA = 90°$. In this triangle, the hypotenuse is R, while the legs have length r and $l/2$. If we can determine θ, we can find R and r using our basic trigonometric relations. If we draw a radius like OX to all the sides of the polygon, we will form $2n$ congruent triangles like OXA. All the angles at O together make up 360°, so $\theta = 360°/2n = 180°/n$.

Thus, we find r and R from the following trigonometric relations:

$$\tan\theta = \tan(180°/n) = \frac{l/2}{r}$$

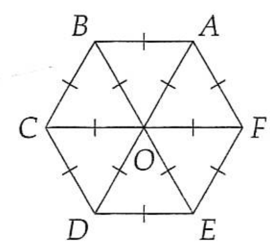
$$\sin\theta = \sin(180°/n) = \frac{l/2}{R}.$$

The area of the polygon is just $2n$ times the area of the OXA, or

$$\text{Area} = 2n[(l/2)(r)/2] = nlr/2.$$

13.4 Regular Hexagons

Regular hexagons appear enough in problems that they merit their own short section. As with many other things which come up often in problems, the main reason hexagons appear so often is that the numbers which pop up in hexagon problems are relatively simple.

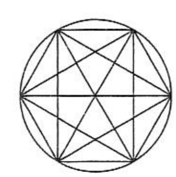


Drawing the lines from the center, O, of the hexagon to the vertices forms six equilateral triangles. (Why?) Chopping regular hexagons into 6 equilateral triangles is in general a good way to attack regular hexagons. For example, it immediately tells us that the area of a regular hexagon with side length s is 6 times the area of an equilateral triangle with side length s, or

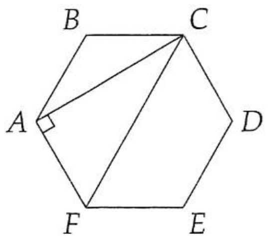
$$[ABCDEF] = 6\left(\frac{s^2\sqrt{3}}{4}\right) = \frac{3s^2\sqrt{3}}{2}.$$

We also see that the longest diagonals, like AD, are twice the length of a side.

EXAMPLE 13-4 Six points are equally spaced around a circle with radius 1. What is the sum of the lengths of all possible segments formed by connecting two of the points?



Solution: As shown, six of the possible segments form sides of a regular hexagon. The center of the circle is the center of the hexagon, so as we saw earlier, the hexagon has side length 1. Three of the segments form main diagonals of the hexagon, and these have length 2.



The other 6 segments are diagonals congruent to AC in the figure at the left. Since $\triangle ABC$ is isosceles and $\angle ABC = 120°$, we have $\angle BAC = 30°$. Since $\angle FAB = 120°$, $\angle FAC$ is a right angle, so $\triangle CAF$ is a 30°-60°-90° triangle. (Make sure you see why this is so.) Thus, $AC = AF\sqrt{3} = \sqrt{3}$. Since there are six such diagonals, six sides, and 3 diagonals like CF, the total of all possible lengths is $6(\sqrt{3}) + 6(1) + 3(2) = \mathbf{12 + 6\sqrt{3}}$.

EXERCISE 13-3 The shortest diagonal of a regular hexagon has length $8\sqrt{3}$. What is the radius of the circle inscribed in the hexagon? (MAΘ 1990)

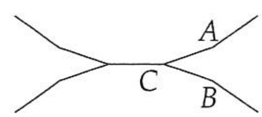
Problems to Solve for Chapter 13

218. Find the number of diagonals that can be drawn in a polygon of 100 sides. (AHSME 1950)

219. Given that $ABCDEF$ is a regular hexagon with side length 6, find the area of triangle BCE. (MATHCOUNTS 1986)

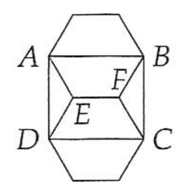
220. Two angles of a convex octagon are congruent. Each of the other angles has a degree measure triple that of each of the first two angles. Find the degree measure of the larger angles. (MAΘ 1990)

221. Two congruent regular 20-sided polygons share a side as shown. Find the degree measure of $\angle ACB$. (MATHCOUNTS 1992)



222. An equilateral triangle and a regular hexagon have equal perimeters. If the area of the triangle is 2, find the area of the hexagon. (AHSME 1970)

223. The coplanar regular hexagons shown share the side EF. Given that the perimeter of quadrilateral $ABCD$ is $44 + 22\sqrt{3}$, find EF. (MATHCOUNTS 1992)



224. Find the area of a regular dodecagon if its circumscribed circle has a circumference of 12π. (MAΘ 1990)

225. Find the ratio of the area of a circle inscribed in a regular hexagon to the area of the circle circumscribed about the same hexagon.

226. Let S be the sum of the interior angles of a polygon P for which each interior angle is 7.5 times the exterior angle at the same vertex. Find S. Is P necessarily regular? (AHSME 1960)

227. A regular polygon with exactly 20 diagonals is inscribed in a circle. The area of the polygon is $144\sqrt{2}$. Find the area of the circle. (MAΘ 1990)

228. Twelve points are equally spaced on the circumference of a circle. How many chords can be drawn that connect pairs of these points and which are longer than the radius of the circle but shorter than its diameter? (MATHCOUNTS 1989)

229. In regular polygon $ABCDE\cdots$, we have $\angle ACD = 120°$. How many sides does the polygon have? (MAΘ 1992)

230. The numbers 1, 2, 3, ..., n are evenly spaced on the rim of a circle. If 15 is directly opposite 49, then find n. (MAΘ 1987)

231. Suppose a goat is tethered to a corner of a building which is in the shape of a regular n-gon. The length of a side of the building and length of the tether are each r. Find the area of the region over which the goat can graze as a function of r and n. (MAΘ 1992)

232. Exactly three of the interior angles of a convex polygon are obtuse. What is the maximum number of sides of such a polygon? (AHSME 1985)

233. A park is in the shape of a regular hexagon 2 km on a side. Starting at a corner, Alice walks along the perimeter of the park for a distance of 5 km. How many kilometers is she from her starting point? (AHSME 1986)

234. If the sum of all the angles except one of a convex polygon is 2190°, then how many sides does the polygon have? (AHSME 1973)

235. Find the sum of angles 1, 2, 3, 4, and 5 in the star-shaped figure shown. (MAΘ 1987)