the Art of Problem Solving

Volume 1: *the Basics*

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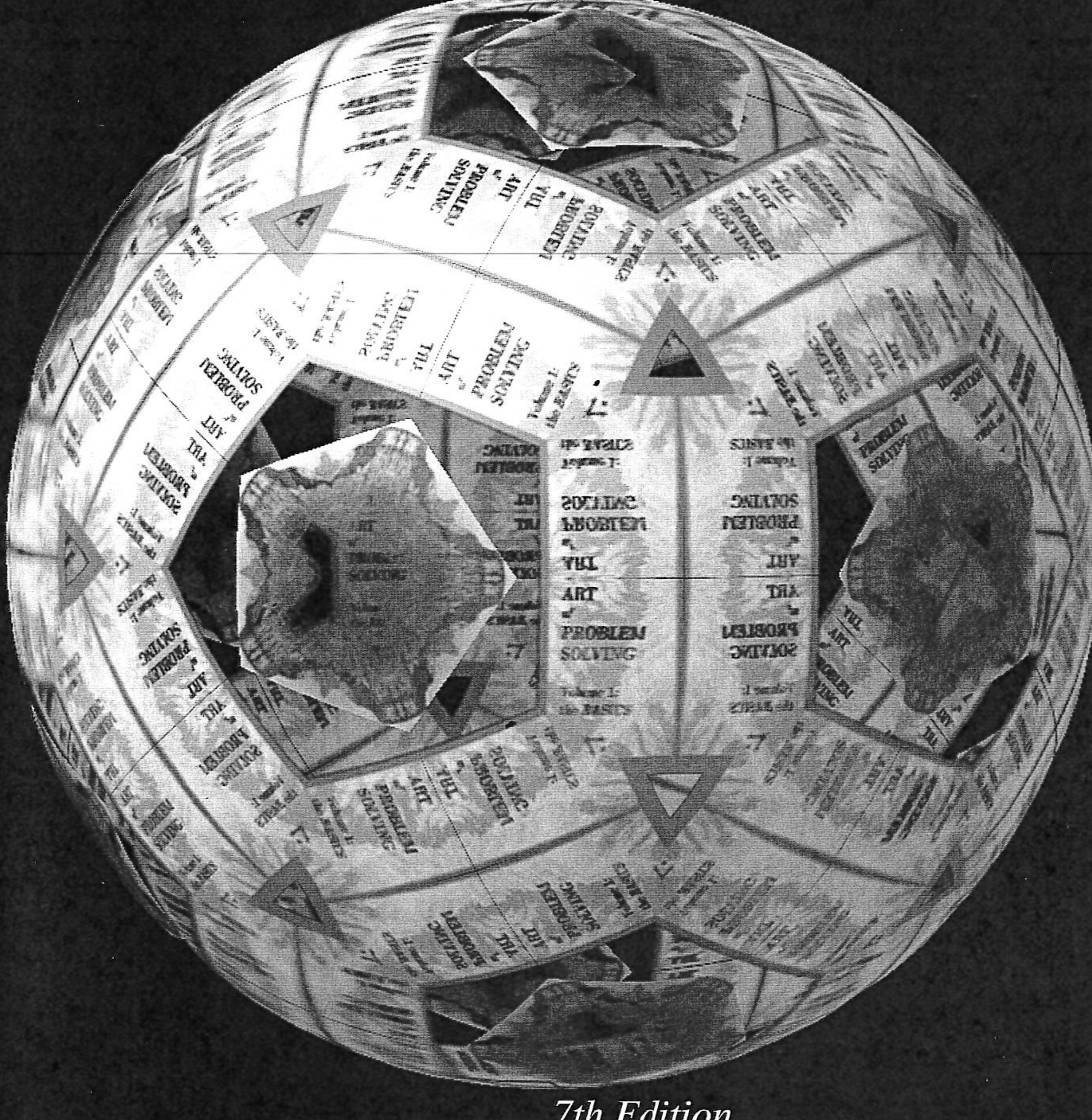

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To Ameyalli, for laughter as clear as your lake, for spirit strong and serene as your skyline, for all that you have taught and learned in four winters. And to Mrs. Wendt, who is still by far the best teacher I ever had.

—SL

For my desert flower Vanessa. Told you we'd make it there eventually.

—RR

Special thanks to the following people who helped make this possible: William and Claire Devlin, Sandor L. and Julieanne G. Lehoczky, Steve and Ann Rubio, Richard and Claire Rusczyk, Stanley Rusczyk.

Thanks

A large number of individuals and organizations have helped make *the ART of PROBLEM SOLVING* possible. All of the following people and groups made very significant contributions, and we offer our deepest gratitude to them all.

Samuel Vandervelde. Sam collaborated with us in creating the Mandelbrot Competition; he continues producing the contest to this day. His work in developing innovative and challenging problems astounds us. In addition to writing these tests, Sam has also contributed problems to the U.S.A. Mathematical Olympiad and created the Stanford Math Circle. Sam is a 1993 graduate of Swarthmore College and earned his Ph.D. in mathematics from the University of Chicago. He was a member of the 1989 U.S. International Mathematics Olympiad team, and was a grader for three years at the Math Olympiad Program, a seminar that determines and prepares that team. Many times when trying to find a proof for some theorem, we'd call on Sam and he'd give us three or four. We owe Sam many thanks for his contributions as a mathematician, our partner, and our friend.

MATHCOUNTS is the premier extracurricular middle school mathematics program in the United States. MATHCOUNTS produces educational problem solving materials and conducts a nationwide contest consisting of school, chapter, state, and national levels. Over 100,000 students participate in the contest each year and hundreds of thousands learn from MATHCOUNTS materials. MATHCOUNTS was the starting point in mathematics for one of the authors, and is a great entry into mathematics for seventh and eighth graders. To Barbara Xhajanka we offer an extra thank you for her help. For more information, visit `www.mathcounts.org`.

The Mandelbrot Competition was started in 1990 by Sam Vandervelde and the authors. It is a four round high school competition designed to teach students not only the common subjects of geometry and algebra, but also subjects that don't appear in high school classes, like number theory and proof techniques. Each round of the Competition consists of an Individual Test and a Team Test. The Individual Test is a short answer test while the Team Test is a series of proofs designed to enhance participants' knowledge of a particular subject area. There are two divisions of the competition, one for beginners and one for more advanced problem solvers. For more information regarding the Mandelbrot Competition, visit `www.mandelbrot.org`.

Dr. George Berzsenyi. We could go on for pages about Dr. Berzsenyi's many contributions to mathematics education through his involvement in competitions and summer programs. He has been involved in writing the AHSME, AIME, and USAMO as well as other independent competitions. He also created the *U.S.A. Mathematical Talent Search* and its international counterpart; participating students in each round are given a month to prepare full solutions to five problems. These solutions are graded by mathematicians and comments on the papers are returned to the students. The

USAMTS is an excellent way for students to learn how to write proofs. The USAMTS is now administered by the Art of Problem Solving Foundation (`www.artofproblemsolving.org`), and is funded primarily by the National Security Agency. For more information on the USAMTS, visit `www.usamts.org`.

Dr. Berzsenyi was also an editor and contributor to the **Mathematics and Informatics Quarterly** (M&IQ). In addition to many practice problems, M&IQ contains articles written (in English) by people all over the world on various subjects of interest to the high school mathematician. While entirely within the reach of the average student, the articles are fascinating and have shown the authors many new approaches to various fields of mathematics.

American Mathematics Competitions. The AMC produces the series of tests that determine the United States mathematics team. The tests are currently called the AMC 10, the AMC 12, the *American Invitational Mathematics Exam* (AIME), the *U.S.A. Junior Mathematical Olympiad* (USAJMO), and the *U.S.A. Mathematical Olympiad* (USAMO). The AMC 12 used to be called the *American High School Mathematics Exam* (AHSME). Top performers in the contests are invited to the Math Olympiad Summer Program (MOP). For more information on the contests and the MOP, visit `amc.maa.org`. There are a handful of problems in this book that appeared on tests at the MOP. These were kindly provided by Professor Cecil Rousseau, who instructed both of the authors of this text at the Math Olympiad Program in 1989.

The **American Regions Mathematics League** (ARML) is an annual competition in which 15-member teams representing schools, cities, and states compete in short answer, proof, and relay contests. The contest is held concurrently at four universities. The authors of this text were teammates on the Alabama team at ARML in 1988 and 1989. We highly recommend this experience to students, as they will learn not only about mathematics but also about teamwork. ARML's primary question writers for the tests from which we have drawn are Gilbert Kessler and Lawrence Zimmerman. For more information on ARML, visit `www.arml.com`.

This text also contains questions from the **Mu Alpha Theta** (MAΘ) National Convention. Mu Alpha Theta is a national high school math honor society. For more information, visit their website at `www.mualphatheta.org`.

Amanda Jones, Maria Monks, and David Patrick. The original *the ART of PROBLEM SOLVING* texts were written in 1993 and 1994 on old Macintosh PCs that have less computing power than most watches now have. To produce the current edition, Amanda Jones recovered these ancient files from our old Macs. Unfortunately, recovering the files was not enough. David Patrick reformatted and edited the book, using his LaTeX expertise to convert our decade-old code to modern LaTeX standards. Finally, all of the diagrams of the book were expertly recreated, and often improved, by Maria Monks.

Key Curriculum Press produces **The Geometer's Sketchpad**, which was used to generate most of the diagrams in the first edition of this text. The Sketchpad is an amazing program that forces students to learn geometry while producing fascinating visual output. The Sketchpad can be used to do everything from teaching simple geometric principles in an interactive way to generating complex fractals. For more information on the Geometer's Sketchpad, visit `www.keypress.com`.

We'd also like to thank **Kai Huang** and **Lauren Williams**, two excellent high school mathematicians, for helping us edit the second edition, and the many members of the online Art of Problem Solving Community at `www.artofproblemsolving.com`, most notably **Ravi and Meena Boppana**, and **Justin Venezuela**, for their corrections for this seventh edition.

To Students

Unless you have been much more fortunate than we were, this book is unlike anything you have used before.

The information in this book cannot be learned by osmosis. What the book teaches is not *facts*, but *approaches*. To learn from a section, you have to read—and comprehend—the text. You will not gain from just looking for the key formulas.

Since subjects are ordered by topic, important ideas may be in seemingly out-of-the-way places, where someone skimming might miss them. Similarly, don't expect to find a uniform difficulty level. When you need to, read slowly, spending minutes on a single line or equation when you need to. Fly when you can. There will be times for both, so don't get impatient.

Some very important concepts are introduced only in examples and exercises. Even when they are simply meant to increase your comfort with the idea at hand, the examples and exercises are the key to understanding the material. Read the examples with even more attention than you pay to the rest of the text, and, no matter what kind of hurry you are in, take the time to do the exercises thoroughly.

This book is about methods. If you find yourself memorizing formulas, you are missing the point. The formulas should become obvious to you as you read, without need of memorization. This is another function of the examples and exercises: to make the methods part of the way you think, not just some process you can remember.

Most of all, this book is about problems. We have gone to great lengths to compile the end-of-chapter problems and other problems in the book. Do them, as many as you possibly can. Don't overload on a single subject, though, or you'll forget everything in a week. Return to each subject every now and then, to keep your understanding current, and to see how much you've grown since you last thought about that subject.

If you have trouble with the problems, don't get neurotic, GET HELP! Consult other students, consult your teachers and, as a last resort, consult the Solution Manual. Don't give up too quickly and begin using the Solution Manual like a text. It should be referred to only after you've made a serious effort on your own. Don't get discouraged. Just as importantly, if these last sentences don't apply to you, you should be the one other students can come to for help.

The book thus comes with one warning: you will not learn if you don't do the problems. Cultivate a creative understanding of the thought processes which go into solving the problems, and before too long you will find you can do them. At that same instant you'll discover that you enjoy them!

To Teachers

This book is our conception of what a student's introduction to elementary mathematics should be.

We strongly feel that a student should learn all subjects simultaneously. There are two reasons for this. First, it helps to convey the interconnectedness of it all; how geometry naturally leads to coordinates and how those coordinates make it easy to define conic sections and the complex plane; how counting leads to probability, the binomial theorem, and number theoretical ideas. Second, it all sinks in better. Overloading on a single subject can cause students to acquire a surface understanding which doesn't connect to any deeper comprehension, and is thus rapidly lost.

You may be surprised at some of the things we do or do not include in this first volume. We put great emphasis on geometry, which we feel is the most neglected subject in many curricula: students take a year of geometry, then don't ever see it again. It is widely felt that American students are very weak in geometry. Other subjects in which our treatments may surprise you are basic number theory and counting, which many would put in an intermediate, rather than an elementary, text. The intuitive appeal of the subjects convinces us that they are excellent introductory material, while other subjects which are less intuitively striking are moved to our second text.

Our notation sometimes diverges from the accepted notation. In these cases, however, our decisions have been made with full deliberation. We strive to use symbols which evoke their meanings, as in the use of the less-popular $\lfloor\ \rfloor$ to denote the greatest integer function instead of the usual [ ].

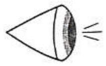
Each chapter of the text is meant to feel like the discussion of a subject with a friend. In one aspect of such a discussion, the text must fail: the answering of questions. This weakness must be repaired by teachers or strong students who are able to assume a leadership role. Teachers are crucial to the process of the book, whether teaching the material directly or simply being available for explanation.

We urge teachers using this in a classroom or club setting to encourage students who understand certain areas to explain the subjects to the rest of the class, or perhaps rotate such responsibility among a large group of willing students. This will not only give the other students a different view, and perhaps one closer to their own thought process, but it also greatly enhances the teaching student's understanding of the subject. Furthermore, the teaching student will have a chance to see the rewards that come from teaching.

We also suggest that after covering each subject, students attempt to write problems using the principles they have learned. In writing a problem, one does much more math than in solving one.

This further inspires the creative drive which is so essential to problem solving in math and beyond, and if students are asked to take a crack at each others' creations, the competitive urge will also be tickled.

In closing, this book is about methods, not memory. The formulas we prove are important ones, but we intend for our explanations to be such that memorization is not necessary. If a student truly understands why a formula is true, then the formula can be internalized without memorization. However you choose to use this book, we hope that the focus remains that students understand *why* formulas work. Only in this way can they understand the full range of the formulas' applications and the full beauty of the mathematics they are learning.



The eye will be found looking at especially important areas of the text. When you see it, pay extra attention.

The threaded needle indicates particularly difficult problems or concepts. If your hands are too shaky, you may need help from someone else.

The bomb signals a warning. If you see it, tread lightly through the material it marks, making sure you won't make the mistakes we warn against.

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