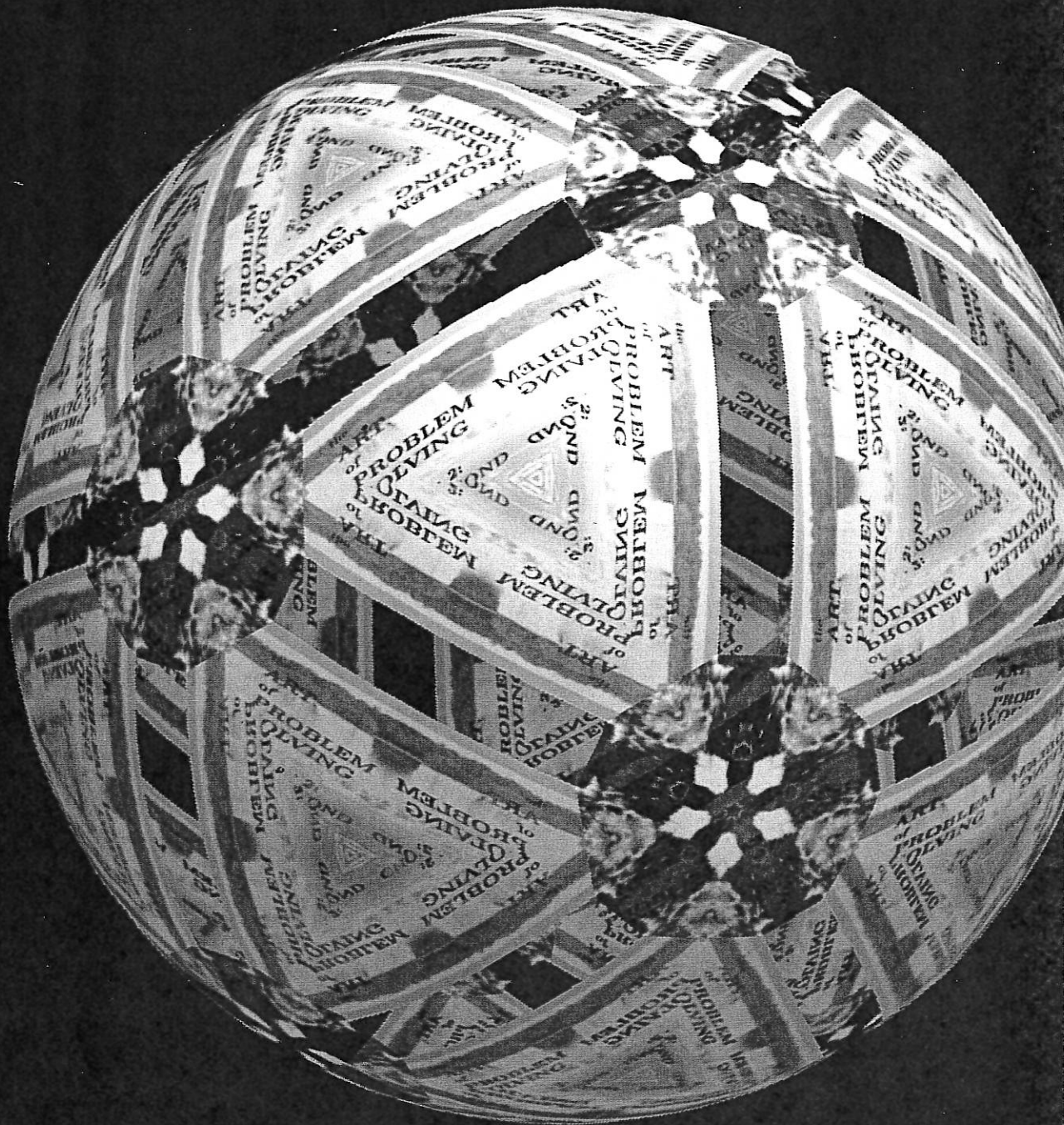


the Art of Problem Solving

Volume 2: and Beyond

Richard Rusczyk
Sandor Lehoczky



7th Edition

the
ART
of
PROBLEM
SOLVING

Volume 2:
and BEYOND

Richard Rusczyk
Sandor Lehoczky

Art of Problem Solving

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The Art of Problem Solving (AoPS) is:

► Books

For over 20 years, the classic *Art of Problem Solving* books have been used by students as a resource for the American Mathematics Competitions and other national and local math events.

Every school should have this in their math library.

– Paul Zeitz, past coach of the U.S. International Mathematical Olympiad team

The Art of Problem Solving Introduction and Intermediate texts form a complete curriculum for outstanding math students in grades 6-12.

The new book [Introduction to Counting & Probability] is great. I have started to use it in my classes on a regular basis. I can see the improvement in my kids over just a short period.

– Jeff Boyd, 4-time MATHCOUNTS National Competition winning coach

► Classes

The Art of Problem Solving offers online classes on topics such as number theory, counting, geometry, algebra, precalculus, calculus, computer programming, and problem solving at beginning, intermediate, and Olympiad levels. Over 8,000 students will participate in an AoPS online class in 2013.

All the children were very engaged. It's the best use of technology I have ever seen.

– Mary Fay-Zenk, coach of National Champion California MATHCOUNTS teams

► Forum

As of November 2013, the Art of Problem Solving Forum has over 130,000 members who have posted over 3,000,000 messages on our discussion board. Members can also participate in any of our free "Math Jams."

I'd just like to thank the coordinators of this site for taking the time to set it up. . . I think this is a great site, and I bet just about anyone else here would say the same. . .

– AoPS Community Member

► Videos, tutorials, interactive resources, and much, much more!

Membership is **FREE**! Come join the Art of Problem Solving community today!

Yalli: I know that somewhere deep down you believe in me, and that makes this worthwhile. I flower dogs.

—SL

Vanessa, thank you for your dedication, support, and patience. Without you, these books would still just be an idea. Thank you for making all my dreams real.

—RR

Thanks

A large number of individuals and organizations have helped make *the ART of PROBLEM SOLVING* possible. All of the following people and groups made very significant contributions, and we offer our deepest gratitude to them all.

Samuel Vandervelde. Sam collaborated with us in creating the Mandelbrot Competition; he continues producing the contest to this day. His work in developing innovative and challenging problems astounds us. In addition to writing these tests, Sam has also contributed problems to the U.S.A. Mathematical Olympiad and created the Stanford Math Circle. Sam is a 1993 graduate of Swarthmore College and earned his Ph.D. in mathematics from the University of Chicago. He was a member of the 1989 U.S. International Mathematics Olympiad team, and was a grader for three years at the Math Olympiad Program, a seminar that determines and prepares that team. Many times when trying to find a proof for some theorem, we'd call on Sam and he'd give us three or four. We owe Sam many thanks for his contributions as a mathematician, our partner, and our friend.

MATHCOUNTS is the premier extracurricular middle school mathematics program in the United States. MATHCOUNTS produces educational problem solving materials and conducts a nationwide contest consisting of school, chapter, state, and national levels. Over 30,000 students participate in the contest each year and hundreds of thousands learn from MATHCOUNTS materials. MATHCOUNTS was the starting point in mathematics for one of the authors, and is a great entry into mathematics for seventh and eighth graders. To Barbara Xhajanka we offer an extra thank you for her help. For more information, visit www.mathcounts.org.

The Mandelbrot Competition was started in 1990 by Sam Vandervelde and the authors. It is a four round high school competition designed to teach students not only the common subjects of geometry and algebra, but also subjects that don't appear in high school classes, like number theory and proof techniques. Each round of the Competition consists of an Individual Test and a Team Test. The Individual Test is a short answer test while the Team Test is a series of proofs designed to enhance participants' knowledge of a particular subject area. There are two divisions of the competition, one for beginners and one for more advanced problem solvers. For more information regarding the Mandelbrot Competition, visit www.mandelbrot.org.

Dr. George Berzsenyi. We could go on for pages about Dr. Berzsenyi's many contributions to mathematics education through his involvement in competitions and summer programs. He has

been involved in writing the AHSME, AIME, and USAMO as well as other independent competitions. He also created the *U.S.A. Mathematical Talent Search* and its international counterpart; participating students in each round are given a month to prepare full solutions to five problems. These solutions are graded by mathematicians and comments on the papers are returned to the students. The USAMTS is an excellent way for students to learn how to write proofs. The USAMTS is now administered by the Art of Problem Solving Foundation (www.artofproblemsolving.org), and is funded primarily by the National Security Agency. For more information on the USAMTS, visit www.usamts.org.

Dr. Berzsenyi was also an editor and contributor to the **Mathematics and Informatics Quarterly** (M&IQ). In addition to many practice problems, M&IQ contains articles written (in English) by people all over the world on various subjects of interest to the high school mathematician. While entirely within the reach of the average student, the articles are fascinating and have shown the authors many new approaches to various fields of mathematics.

American Mathematics Competitions. The AMC produces the series of tests that determine the United States mathematics team. The tests are currently called the AMC 10, the AMC 12, the *American Invitational Mathematics Exam* (AIME), and the *U.S.A. Mathematical Olympiad* (USAMO). The AMC 12 used to be called the *American High School Mathematics Exam* (AHSME). Top performers in the contests are invited to the Math Olympiad Summer Program (MOP). For more information on the contests and the MOP, visit amc.maa.org. There are a handful of problems in this book that appeared on tests at the MOP. These were kindly provided by Professor Cecil Rousseau, who instructed both of the authors of this text at the Math Olympiad Program in 1989.

The **American Regions Mathematics League** (ARML) is an annual competition in which 15-member teams representing schools, cities, and states compete in short answer, proof, and relay contests. The contest is held concurrently at multiple sites. The authors of this text were teammates on the Alabama team at ARML in 1988 and 1989. We highly recommend this experience to students, as they will learn not only about mathematics but also about teamwork. ARML's primary question writers for the tests from which we have drawn are Gilbert Kessler and Lawrence Zimmerman. For more information on ARML, visit www.arml.com.

David Patrick, Amanda Jones, and Naoki Sato. The original *the ART of PROBLEM SOLVING* texts were written in 1993 and 1994 on old Macintosh PCs that have less computing power than most watches now have. To produce the current edition, Amanda Jones recovered these ancient files from our old Macs. Unfortunately, recovering the files was not enough. David Patrick reformatted and edited the book, using his \LaTeX expertise to convert our decade-old code to modern \LaTeX standards. Finally, nearly all of the diagrams of the book were re-created by Richard Rusczyk, Naoki Sato, and Amanda Jones.

This text also contains questions from the Mu Alpha Theta ($\text{MA}\Theta$) National Convention. Mu Alpha Theta is a national high school math honor society. For more information, visit their website at www.mualphatheta.org.

We gathered some problems from a few international sources in order to offer a wealth of challenging problems on some advanced topics. We collected problems from the national olympiads of **Bulgaria** (provided by Borislav Lazarov) and **Canada** (provided by Graham Wright). Both of these sources provide excellent practice for problem solvers. We also include problems that were either used in or proposed for the **International Mathematical Olympiad** (IMO). Each year many of the countries in the world send a six person team to the IMO to participate in the Olympiad. The

problems in this text come from the 1989 Olympiad in Germany (provided by Paul Jainta), and the 1986 and 1985 Olympiads in Poland and Finland respectively (provided by Dr. George Berzsenyi).

Key Curriculum Press produces **The Geometer's Sketchpad**, which was used to generate most of the diagrams in the first edition of this text. The Sketchpad is an amazing program that forces students to learn geometry while producing fascinating visual output. The Sketchpad can be used to do everything from teaching simple geometric principles in an interactive way to generating complex fractals. For more information on the Geometer's Sketchpad, visit www.keypress.com.

Special thanks to Vanessa Ruczyk and Vladmir Vukicevic for their help in proofreading this book and to Kai Huang, Joon Pahk, Lauren Williams, and many members of the online Art of Problem Solving Community at www.artofproblemsolving.com, and particularly Hussain Zahid Sheikh, Ravi and Meena Boppana, and Justin Venezuela, for their corrections for this seventh edition.

To Students

Unless you have been much more fortunate than we were, this book is unlike anything you have used before (except Volume 1!).

The information in this book cannot be learned by osmosis. What the book teaches is not *facts*, but *approaches*. To learn from a section, you have to read, and comprehend, the text. You will not gain from just looking for the key formulas.

Important ideas may be in seemingly out-of-the-way places, where someone skimming might miss them, since things are ordered by topic, not by importance. Don't expect to find a uniform difficulty level. Read slowly, spending minutes on a single line or equation when you need to. Fly when you can. There will be times for both, so don't get impatient.

Some very important concepts are introduced only in examples and exercises. Even when they are simply meant to increase your comfort with the idea at hand, the examples and exercises are the key to understanding the material. Read the examples with even more attention than you pay to the rest of the text, and, no matter what kind of hurry you are in, take the time to do the exercises thoroughly.

This book is about methods. If you find yourself memorizing formulas, you are missing the point. The formulas should become obvious to you as you read, without need of memorization. This is another function of the examples and exercises: to make the methods part of the way you think, not just some process you can remember.

The subjects in this volume cover a much broader range of difficulty than those in Volume 1; therefore, you may wish to do a lot of skipping around. If you hit a subject you simply don't understand, move on and return later when you've had more practice problem solving. Don't give up; learning takes time.

Most of all, this book is about problems. We have gone to great lengths to compile the end-of-chapter problems and other problems in the book. Do them, as many as you possibly can. Don't overload on a single subject, though, or you'll forget everything in a week. Return to each subject every now and then, to keep your understanding current, and to see how much you've grown since you last thought about that subject.

If you have trouble with the problems, don't get neurotic, GET HELP! Consult other students, consult your teachers and, as a last resort, consult the Solution Manual. Don't give up too quickly

and begin using the Solution Manual like a text. It should be referred to only after you've made a serious effort on your own. Don't get discouraged. Just as importantly, if these last sentences don't apply to you, you should be the one other students can come to for help.

The book thus comes with one warning: you will not learn if you don't do the problems. Cultivate a creative understanding of the thought processes which go into solving the problems, and before too long you will find you can do them. At that same instant you'll discover that you enjoy them!

To Teachers

the ART of PROBLEM SOLVING is our conception of what a motivated student's instruction in high school non-calculus mathematics should be.

We strongly feel that a student should learn all subjects simultaneously. There are two reasons for this. First, it is better to convey the interconnectedness of it all; how geometry naturally leads to coordinates and how those coordinates make it easy to define conic sections and the complex plane; how counting leads to probability, the Binomial Theorem, and number theoretical ideas. Second, it all sinks in better. Overloading on a single subject can cause students to acquire a surface understanding which doesn't connect to any deeper comprehension, and is thus rapidly lost.

There are many subjects in this text which your students have likely not seen before. We feel it is very unfortunate that students aren't introduced to such subjects as collinearity, inequalities, and number theory. Again in this volume we put an emphasis on geometry, which we feel is the most neglected subject in many curricula: students take a year of geometry, then don't ever see it again.

We also warn the teacher that the difficulty level of the subjects in this book vary much more greatly than in the first volume. Some of the text may be too advanced for your beginners, while other portions are likely too elementary for your advanced students. Thus, take care in the chapters or sections you assign your students.

Our notation sometimes diverges from the accepted notation. In these cases, however, our decisions have been made with full deliberation. We strive to use symbols which evoke their meanings, as in the use of the less-popular $\lfloor \rfloor$ to denote the greatest integer function instead of the usual $[]$.

Each chapter of the text is meant to feel like the discussion of a subject with a friend. In one aspect of such a discussion, the text must fail: the answering of questions. This weakness must be repaired by teachers or strong students who are able to assume a leadership role. Teachers are crucial to the process of the book, whether teaching the material directly or simply being available for explanation.

We urge teachers using *the ART of PROBLEM SOLVING* in a classroom or club setting to encourage students who understand certain areas to explain the subjects to the rest of the class, or perhaps rotate such responsibility among a large group of willing students. This will not only give the other students a different view, perhaps closer to their own thought process, but it also greatly

enhances the teaching student's understanding of the subject. Furthermore, the instructing student will have a chance to see the rewards that come from teaching.

We also suggest that after covering each subject, students attempt to write problems using the principles they have learned. In writing a problem, one does much more math than in solving one. This further inspires the creative drive which is so essential to problem solving in math and beyond, and if students are asked to take a crack at each others' creations, the competitive urge will also be tickled.

In closing, this book is about methods, not memory. The formulas we prove are important ones, but we intend for our explanations to be such that memorization is not necessary. If a student truly understands why a formula is true, then the formula can be internalized without memorization. However you choose to use this book, we hope that the focus remains that students understand why formulas work. Only in this way can they understand the full range of the formulas' applications and the full beauty of the mathematics they are learning.

Justify Your Love

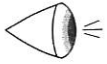
Throughout high school and even middle school, the authors of this text participated in a lot of math contests. After high school we then produced our own contest, The Mandelbrot Competition, along with Sam Vandervelde. One question has persisted from the skeptics: why bother? They argue that the math involved in competitions is largely useless for the rest of participants' lives. While correct (It won't be often that your boss says, "Tell me $\phi(45)$ or you're fired!"), this argument is misguided, because math is by far not the most important aspect of the contests.

Through math competitions and projects, students learn how to attack problems. Unlike specific techniques, this skill is crucial to virtually any area of life. Successful problemists go on to be successful not only in mathematics, but also in every other field (not just technical ones!) that you can think of. The authors' math training didn't just make us able to write this text, but it taught us the rewards of hard work, gave us confidence, and—most importantly—developed our ability to solve problems.

Good problemists are very creative people. Knowing all the tools at your disposal will not always guarantee finding a solution; the key to solving problems is cleverly choosing the right method of attack. A great way to 'train' for problem solving is to do various brainstorming and other creative ideas. Not only will these help you open your eyes to new ideas, but they can often be a lot of fun.

This is not to say that the mathematics itself is useless. Hopefully through this text and other work, you'll develop the same interest in mathematics we have. While some people might think we're nuts, we view an elegant mathematical concept or a neat proof with the same admiration as others view a Rembrandt painting or a Beethoven symphony. This is the reason for our choices of the covers of our texts. The beauty of nature is dictated by a mathematics which we humans are still struggling to understand.

The last and, for many, most important aspect of math contests is the people. The authors of this text met each other and Sam, as well as many other friends, through math. When your days in contests are over, you'll cherish the memories far more than you will the contests themselves.



The eye will be found looking at especially important areas of the text. When you see it, pay extra attention.



The threaded needle indicates particularly difficult problems or concepts. If your hands are too shaky, you may need help from someone else.



The bomb signals a warning. If you see it, tread lightly through the material it marks, making sure you won't make the mistakes we warn against.

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